Methods of Applied Mathematics

Sheet 5 solutions

1. a)
$$J(y) = \int_0^1 y' dx, \quad y(0) = 0, \quad y(1) = 1.$$
Since $L = y'$, Euler equation is
$$-\frac{d}{dx}(1) = 0$$
this is identically satisfied and so tells us nothing.

b)
$$J(y) = \int_0^1 yy' dx, \quad y(0) = 0, \quad y(1) = 1.$$
Since $L = yy'$, Euler equation is
$$y' - \frac{d}{dx}(y) = 0$$
this is again identically satisfied and so tells us nothing.

c)
$$J(y) = \int_0^1 xyy' dx, \quad y(0) = 0, \quad y(1) = 1.$$
Since $L = xyy'$, Euler equation is
$$xy' - \frac{d}{dx}(xy) = 0 \quad \Rightarrow \quad xy' - y - xy' = 0$$
$$\Rightarrow y = 0.$$
This does not satisfy $y(1) = 1$.

2.
$$J(y) = \int_a^b (x^2 y'^2 + y^2) dx,$$
Since $L = x^2 y'^2 + y^2$, Euler equation is
$$2y - \frac{d}{dx}(2x^2 y') = 0 \quad \Rightarrow \quad 2y - 4xy' - 2x^2 y'' = 0$$
$$\Rightarrow x^2 y'' + 2xy' - y = 0.$$
This is an ODE of Eluer type: for a solution try $y = x^k$.
So $y' = kx^{k-1}, \; y'' = k(k-1)x^{k-2}$ an substituting into the D.E. gives
$$k(k-1) + 2k - 1 = 0 \quad \Rightarrow \quad k^2 + k - 1 = 0$$
$$\Rightarrow \left(k + \frac{1}{2}\right)^2 - \frac{5}{4} = 0 \;\Rightarrow\; k = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$
So general solution is
$$y = x^{-1/2}(Ax^{\sqrt{5}/2} + Bx^{-\sqrt{5}/2}).$$
The coefficinets A and B are given by the values at the fixed end points.

3. **Theorem:**

If a function y gives an extreme value of the functional

$$J(y) = \int_a^b L(x, y, y')dx$$

for $y \in F = \{y \in C^2[a,b]\}$, then y satisfies the Euler equation

$$L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y') = 0,$$

and the boundary conditions at a and b are

$$L_{y'}(a, y(a), y'(a)) = 0, \;\; L_{y'}(b, y(b), y'(b)) = 0.$$

holds at $x = b$.

Proof:

As for the case in which both end points are fixed, we consider $V'(0) = 0$ where

$$V(\varepsilon) = \int_a^b L(x, y + \varepsilon h, y' + \varepsilon h')dx$$

with $h \in C^2[a,b]$, but now without any conditions on $h(a)$ or $h(b)$. As in the proof of the theorem for fixed end points we have

$$V'(0) = 0 \Leftrightarrow \int_a^b \{L_y(x, y, y')h(x) + L_{y'}(x, y, y')h'(x)\} \, dx = 0.$$

After in integrating by parts we have

$$\int_a^b \{L_y(x, y, y')h(x) + L_{y'}(x, y, y')h'(x)\} \, dx = 0$$

$$\Leftrightarrow \int_a^b \left\{ L_y(x, y, y') + \frac{d}{dx} L_{y'}(x, y, y') \right\} h(x)dx + [L_{y'}(x, y, y')h(x)]_a^b = 0 \qquad (1)$$

A necessary condition for an extremal is that this should hold for all admissible h. In particular it should hold for $h(a) = h(b) = 0$. Hence we get that

$$[L_{y'}(x, y, y')h(x)]_a^b = 0$$

and thus

$$\int_a^b \left\{ L_y(x, y, y') + \frac{d}{dx} L_{y'}(x, y, y') \right\} h(x)dx = 0.$$

The fundamental lemma of calculus of variations then gives

$$L_y(x, y, y') + \frac{d}{dx} L_{y'}(x, y, y') = 0.$$

So returning to (1) we have

$$[L_{y'}(x, y, y')h(x)]_a^b = 0 \qquad \text{for all admissible } h.$$

In particular this holds for h with

$$h(a) \neq 0 \;\text{ and }\; h(b) = 0 \;\Rightarrow\; L_{y'}(a, y(a), y'(a)) = 0.$$

It also holds for h with $h(a) = 0$ and $h(b) \neq 0$

$$\Rightarrow L_{y'}(b, y(b), y'(b)) = 0.$$

4. a)

$$J(y) = \int_0^1 (y'^2 + y^2)dx \quad y(0) = 1,\ y(1) \text{ unspecified.}$$

The Lagrangian is $L = y'^2 + y^2$ so the Euler equation is

$$2y - \frac{d}{dx}(2y') = 0 \ \Rightarrow \ y'' - y = 0$$
$$\Rightarrow \ y = A\cosh x + B\sinh x.$$

The boundary condition $y(0) = 1$ gives

$$1 = A$$

and hence we have

$$y = \cosh x + B\sinh x.$$

The free end-point condition is

$$L_{y'}(1, y(1), y'(1)) = 0 \Leftrightarrow \ 2y'(1) = 0 \ \Rightarrow \ 0 = \sinh 1 + B\cosh 1$$
$$\Rightarrow B = -\tanh 1.$$

So

$$y = \cosh x - \tanh 1\sinh x.$$

b)

$$J(y) = \int_0^1 (y'^2 + 2y'y + 2y' + y)dx \quad y(0) = 2,\ y(1) \text{ unspecified.}$$

The Lagrangian is $L = y'^2 + 2y'y + 2y' + y$ so the Euler equation is

$$2y' + 1 - \frac{d}{dx}((2y' + 2y + 2) = 0 \ \Rightarrow \ 2y'' - 1 = 0$$
$$\Rightarrow \ y = \frac{1}{4}x^2 + Ax + B.$$

Since $y(0) = 2$ we have $B = 2$ and so

$$y = 2 + \frac{1}{4}x^2 + Ax.$$

The free end-point condition is

$$L_{y'}(1, y(1), y'(1)) = 0 \Leftrightarrow \ 2y'(1) + 2y(1) + 2 = 0 \ \Rightarrow 2(1/2 + A) + 2(2 + 1/4 + A) + 2 = 0$$
$$\Rightarrow 1 + 2A + 4 + 1/2 + 2A + 2 = 0 \ \Rightarrow \ A = -15/8.$$

So

$$y = \frac{1}{4}x^2 - \frac{15}{8}x + 2.$$

5. The Euler equation is

$$L_y - \frac{d}{dx}(L_{y'}) + \frac{d^2}{dx^2}(L_{y''}) = 0$$

with $L = 1 + y''^2$. This gives

$$0 - 0 + \frac{d^2}{dx^2}(2y'') = 0 \ \Rightarrow \ y'''' = 0$$
$$\Rightarrow \ y = A + Bx + Cx^2 + Dx^3.$$

Since

$$y' = B + 2Cx + 3Dx^2$$

the boundary conditions give

$$A = 0,\ B = 0,\ A + B + C + D = 1,\ B + 2C + 3D = 1$$

which yield

$$A = 0,\ B = 0,\ C = 2,\ D = -1$$

and so the extremal is

$$y = 2x^2 - x^3.$$