Methods of Applied Mathematics

Sheet 4 solutions

1. The direct method puts $x = x_0 + \varepsilon x_1 + \cdots$ and substitutes this into the equation to get

$$\varepsilon(x_0 + \varepsilon x_1 + \cdots)^3 + (x_0 + \varepsilon x_1 + \cdots) - 2 = 0$$

Coefficient of ε^0: $x_0 - 2 = 0 \Rightarrow x_0 = 2$.

Coefficient of ε^1: $x_0^3 + x_1 = 0 \Rightarrow x_1 = -x_0^3 = -8$.

This gives the root near 2 to be $2 - 8\varepsilon$ up to first order correction.

Now rescale, using $\bar{x} = x/\delta$ to get

$$\varepsilon\delta^3\bar{x}^3 + \delta\bar{x} - 2 = 0.$$

Possible balances are

$$(a) \quad \varepsilon\delta^3 \sim \delta \quad \Rightarrow \quad \delta \sim 1/\sqrt{\varepsilon}$$

and

$$(b) \quad \varepsilon\delta^3 \sim 1 \quad \Rightarrow \quad \delta \sim \varepsilon^{-1/3}.$$

$$(c) \quad \delta \sim 1 \quad \Rightarrow \text{ no scaling and hence reject.}$$

First we try (b) and hence we set $\delta = 1/\sqrt{3}\varepsilon$ to get

$$\bar{x}^3 + \frac{1}{\sqrt{3}\varepsilon}\bar{x} - 2 = 0 \;\Rightarrow\; \sqrt{3}\varepsilon\bar{x}^3 + \bar{x} - 2\sqrt{3}\varepsilon = 0.$$

Setting $\bar{x} = \bar{x}_0 + \sqrt{3}\varepsilon\bar{x}_1 + (\sqrt{3}\varepsilon)^2\bar{x}_2 = \cdots$ gives the following zeroth order approximation

$$\bar{x}_0 = 0$$

we reject this choice as gives an approximate solution this is not of moderate size.

Now we try (a) and put $\delta = 1/\sqrt{\varepsilon}$ to get

$$\frac{1}{\sqrt{\varepsilon}}\bar{x}^3 + \frac{1}{\sqrt{\varepsilon}}\bar{x} - 2 = 0 \;\Rightarrow\; \bar{x}^3 = \bar{x} - 2\sqrt{\varepsilon} = 0.$$

Now take $\bar{x} = \bar{x}_0 + \sqrt{\varepsilon}\bar{x}_1 + (\sqrt{\varepsilon})^2\bar{x}_2 = \cdots$ to get

$$(\bar{x}_0 + \sqrt{\varepsilon}\bar{x}_1 + \cdots)^3 + (\bar{x}_0 + \sqrt{\varepsilon}\bar{x}_1 + \cdots) - 2\sqrt{\varepsilon} = 0.$$

Coefficient $(\sqrt{\varepsilon})^0$: $\bar{x}_0^3 + \bar{x}_0 = 0 \Rightarrow \bar{x}_0 = \pm i$.

Coefficient $(\sqrt{\varepsilon})^1$: $3\bar{x}_0^2\bar{x}_1 + \bar{x}_1 - 2 = 0 \Rightarrow \bar{x}_1 = \frac{2}{3\bar{x}_0^2+1} = -1$.

So (to first order), $\bar{x} = \pm i - \sqrt{\varepsilon}$. In terms of original variables this gives $\pm\frac{i}{\sqrt{\varepsilon}} - 1$ for the other two roots.

Up to order retained, the product of the roots is

$$(2 - 8\varepsilon)(-1 + \frac{i}{\sqrt{\varepsilon}})(-1 - \frac{i}{\sqrt{\varepsilon}}) = (2 - 8\varepsilon)(1 + \frac{1}{\varepsilon}) = 2 + \frac{2}{\varepsilon} - 8\varepsilon - 8 = \frac{2}{\varepsilon} - 6 - 8\varepsilon.$$

Since the product of the roots of a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ should be equal to a_0/a_n we see that this is true to leading order since $a_0/a_n = 2/\varepsilon$.

2. (a)

$$\varepsilon\ddot{x} + 2\dot{x} + e^x = 0, \quad 0 < \varepsilon << 1, \; x(0) = 0, \; x(1) = 0$$

Direct method gives

$$2\dot{x}_0 + e^{x_0} = 0,\ x_0(0) = 0,\ x_0(1) = 0$$

as $e^{x_0+\varepsilon x_1+\cdots} = e^{x_0}e^{\varepsilon x_1+\cdots} = e^{x_0}(1+\cdots)$.

So

$$2e^{-x_0}\dot{x}_0 + 1 = 0 \Rightarrow 2e^{-x_0}\frac{dx_0}{dt} = -1 \Rightarrow \int 2e^{-x_0}dx_0 = -\int dt$$

$$\Rightarrow -2e^{-x_0} + t = A \Rightarrow x_0 = -\ln\left(\frac{t-A}{2}\right).$$

$A = -1$ gives $x_0(1) = 0$ but then $x_0(0) = 0$ is not satisfied. Assume that there is a boundary layer at 0 and that we have found

$$x_{\text{outer}}(t) = \ln\frac{2}{t+1}.$$

Now rescale using $s = t/\delta$ as independent variable. The equation becomes

$$\frac{\varepsilon}{\delta^2}x'' + \frac{2}{\delta}x' + e^x = 0 \quad \text{dash} = \frac{d}{ds}$$

with $x(0) = 0$ as boundary condition. Possible balances are

$$(a) \quad \frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \delta \sim \varepsilon$$

$$(b) \quad \frac{\varepsilon}{\delta^2} \sim 1 \Rightarrow \delta \sim \sqrt{\varepsilon}$$

$$(c) \quad \frac{2}{\delta} \sim 1 \Rightarrow \delta \sim 1 \Rightarrow \text{no scaling and hence reject.}$$

First we try (b) and set $\delta = \sqrt{\varepsilon}$ to get

$$x'' + \frac{2}{\sqrt{\varepsilon}}x' = e^x$$

$$\Rightarrow \sqrt{\varepsilon}x'' + 2x' = \sqrt{\varepsilon}e^x.$$

We reject this choice as it gives a reduction in order for the equation satisfied by the zeroth order approximation.

Now we try (a) and put $\delta = \varepsilon$ to get

$$x'' + 2x' = \varepsilon e^x.$$

Zeroth order approximation satisifies

$$x_0'' + 2x_0' = 0,\ x_0(0) = 0$$

General solution is $x_0(s) = A + Be^{-2s}$ and the boundary condition implies $B = -A$. So we have

$$x_{\text{inner}}(s) = A(1 - e^{-2s}).$$

$$\Rightarrow \quad x_{\text{inner}}(t) = A(1 - e^{-2t/\varepsilon}).$$

For matching use the intermediate variable $u = t/\sqrt{\varepsilon}$. So $t = u\sqrt{\varepsilon}$ and

$$x_{\text{outer}} = \ln\left(\frac{2}{u\sqrt{\varepsilon}+1}\right), \quad x_{\text{inner}} = A(1 - e^{-2u/\sqrt{\varepsilon}}).$$

For a match we need

$$\lim_{\varepsilon\downarrow 0} x_{\text{outer}} = \lim_{\varepsilon\downarrow 0} x_{\text{inner}} = \text{ common limit} \qquad \Rightarrow \ln 2 = A = \text{ common limit.}$$

So the approximation on $[0,1]$ is

$$\begin{aligned} x_a(t) &= x_{\text{outer}} + x_{\text{inner}} - \text{ common limit} \\ &= \ln\frac{2}{t+1} + \ln 2\{1-e^{-2t/\varepsilon}\} - \ln 2 \\ &= \ln\frac{2}{t+1} - e^{-2t/\varepsilon}\ln 2. \end{aligned}$$

(b)

$$\varepsilon\ddot{x} + \dot{x} = 2(1-t), \quad 0 < \varepsilon << 1, \; x(0) = 1, \; x(1) = 1$$

Direct method gives

$$\dot{x}_0 = 2(1-t), \; x_0(0) = 1, \; x_0(1) = 1$$

for the zeroth order approximation.

The solution of the ODE is

$$x_0 = 2t - t^2 + A.$$

Using the boundary condition $x_0(1) = 1$ gives $A = 0$ and hence $x_0 = 2t - t^2$ which does not satisfy $x_0(0) = 1$.

Assume we have an outer solution

$$x_{\text{outer}}(t) = 2t - t^2$$

and look for an inner solution valid near $t = 0$ by rescaling the independent variable: $s = t/\delta$.

The equation becomes

$$\frac{\varepsilon}{\delta^2}x'' + \frac{1}{\delta}x' = 2 - 2\delta s \quad \text{dash} = \frac{d}{ds}$$

with $x(0) = 1$ as boundary condition. There are 4 coefficients $\frac{\varepsilon}{\delta^2}$, $\frac{1}{\delta}$, 2, 2δ and hence 6 possible balances are

$$\begin{aligned} &(a) \quad \frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \;\Rightarrow\; \delta \sim \varepsilon \\ &(b) \quad \frac{\varepsilon}{\delta^2} \sim 2 \;\Rightarrow\; \delta \sim \sqrt{\varepsilon} \\ &(c) \quad \frac{\varepsilon}{\delta^2} \sim 2\delta \;\Rightarrow\; \delta \sim \varepsilon^{1/3} \\ &(d) \quad \frac{1}{\delta} \sim 2 \;\Rightarrow\; \delta \sim 1 \\ &(e) \quad \frac{1}{\delta} \sim 2\delta \;\Rightarrow\; \delta \sim 1 \\ &(f) \quad 2 \sim 2\delta \;\Rightarrow\; \delta \sim 1 \end{aligned}$$

We reject (d), (e) and (f) and try (a), (b) and (c) as possible choices.

Possibility (b): Setting $\delta = \sqrt{\varepsilon}$ gives

$$x'' + \frac{1}{\sqrt{\varepsilon}}x' = 2 - 2\sqrt{\varepsilon}s$$

$$\Rightarrow \; \sqrt{\varepsilon}x'' + x' = 2\sqrt{\varepsilon} - 2\varepsilon s$$

This gives $x_0' = 0$ which we reject as it gives a reduction in order for the zeroth order approximation.

Possibility (c): Setting $\delta = \varepsilon^{1/3}$ gives

$$\varepsilon^{1/2}x'' + \frac{1}{\varepsilon^{1/3}}x' = 2 - 2\varepsilon s$$

$$\Rightarrow \; \varepsilon^{2/3}x'' + x' = 2\varepsilon^{1/3} - 2\varepsilon^{2/3}s$$

This also gives $x_0' = 0$ so we reject it too.

Possibility (a): Setting $\delta = \varepsilon$ gives

$$\frac{1}{\varepsilon}x'' + \frac{1}{\varepsilon}x' = 2 - 2\varepsilon s$$
$$\Rightarrow \; x'' + x' = 2\varepsilon - 2\varepsilon^2 s$$
$$\Rightarrow x_0'' + x_0' = 0 \quad \text{which looks o.k.}$$

General solution is $x_0(s) = A + Be^{-s}$ and the boundary condition $x_0(0) = 1$ implies $A + B = 1$ i.e. $B = 1 - A$ so

$$x_0(s) = A + (1 - A)e^{-s}$$
$$\Rightarrow \; x_0(t) = A + (1 - A)e^{-t/eps}$$

For matching we use the intermediate varaible $u = \sqrt{s}t = t/\sqrt{\varepsilon}$ and require that

$$\lim_{\varepsilon\downarrow 0} x_{\text{outer}} = \lim_{\varepsilon\downarrow 0} x_{\text{inner}} = \text{ common limit}$$

$$\Rightarrow \lim_{\varepsilon\downarrow 0}(2u\sqrt{\varepsilon} - \varepsilon u^2) = \lim_{\varepsilon\downarrow 0}(A + (1 - A)e^{-u/\sqrt{\varepsilon}}) \quad \Rightarrow 0 = A.$$

So the common limit is 0 and

$$\begin{aligned} x_a(t) &= x_{\text{outer}} + x_{\text{inner}} - \text{ common limit} \\ &= 2t - t^2 + e^{-t/\varepsilon}. \end{aligned}$$

3.

$$\varepsilon\ddot{x} - \dot{x} = 2t, \; x(0) = 1, \; x(1) = 1$$

The direct method yields

$$-\dot{x}_0 = 2t, \; x_0(0) = 1, \; x_0(1) = 1.$$

Proceeding as in the lectures (i.e. using the boundary condition at $t = 1$ and assuming a boundary layer at $t = 0$) gives

$$x_{\text{outer}} = 2 - t^2.$$

For the inner solution, rescale using $s = t/\delta$. The equation becomes

$$\frac{\varepsilon}{\delta^2}x'' - \frac{1}{\delta}x' = 2\delta s \quad (\text{dash} = \frac{d}{ds})$$

with $x(0) = 1$.

There are 3 possible balances:

$$(a) \quad \frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \;\Rightarrow\; \delta \sim \varepsilon \quad \text{possible}$$
$$(b) \quad \frac{\varepsilon}{\delta^2} \sim 2\delta \;\Rightarrow\; \delta \sim \varepsilon^{1/3} \quad \text{possible}$$
$$(c) \quad \frac{1}{\delta} \sim 2\delta \;\Rightarrow\; \delta \sim 1 \quad \text{reject}$$

Possibility (b) implies (putting $\delta = \varepsilon^{1/3}$)

$$\varepsilon^{1/3}x'' - \frac{1}{\varepsilon^{1/3}}x' = 2\varepsilon^{1/3}s$$

$$\Rightarrow \varepsilon^{2/3}x'' - x' = 2\varepsilon^{2/3}s$$

$\Rightarrow -x_0' = 0$ which is first order (and hence is a reduction in order) so reject (b).

Possibility (a) implies (putting $\delta = \varepsilon$)

$$\frac{1}{\varepsilon}x'' - \frac{1}{\varepsilon}x' = 2\varepsilon s$$
$$\Rightarrow x'' - x' = 2\varepsilon^2 s \;\Rightarrow x_0'' - x_0' = 0.$$

This is second order, so looks o.k. and has general solution

$$x_0(s) = A + Be^s.$$

The boundary condition $x(0) = 1$ implies $A + B = 1 \;\Rightarrow\; B = 1 - A$, so

$$x_0(s) = A + (1 - A)e^s$$
$$\Rightarrow\; x_0(t) = A + (1 - A)e^{t/\varepsilon}$$

For matching we use the intermediate variable

$$u = \sqrt{s}t = t/\sqrt{\varepsilon}$$

and the condition that

$$\lim_{\varepsilon\downarrow 0} x_{\text{outer}} = \lim_{\varepsilon\downarrow 0} x_{\text{inner}} = \text{common limit}$$
$$\Rightarrow \lim_{\varepsilon\downarrow 0}(2 - \varepsilon u^2) = \lim_{\varepsilon\downarrow 0}(A + (1 - A)e^{\frac{u}{\sqrt{\varepsilon}}}) = \text{ common limit.}$$

This fails as the limit on the left is finite and the one on the right is infinite.

The assumption that the boundary layer is at $t = 0$ is incorrect. If we replace the independent variable by $\tilde{t} = 1 - t$, this has the effect of swapping the end points so that $t = 1$ becomes $\tilde{t} = 0$. The problem becomes

$$\varepsilon\ddot{x} + \dot{x} = 2(1 - \tilde{t}), \quad x(0) = 1,\; x(1) = 1,$$

where $\tilde{t}$ is now the independent variable and dots denote $d/d\tilde{t}$. This is the problem of Ex. 2(b) and was discussed above.

4. a)

$$\varepsilon\ddot{x} + (t + 1)\dot{x} + x = 0,\; x(0) = 0,\; x(1) = 1.$$

Comparison with theorem gives $p(t) = t + 1$ and $q(t) = 1$ so p and q are cntinous on $[0, 1]$ with $p(t) > 0$ for $t \in [0, 1]$ as required.
The theorem gives

$$\begin{aligned} x_{\text{outer}} &= \exp\left(\int_t^1 \frac{d\tau}{\tau + 1} d\tau\right) = \exp([\ln(\tau + 1)]_t^1) \\ &= \exp\ln\left(\frac{2}{t + 1}\right) = \frac{2}{t + 1}. \end{aligned}$$

and $x_{\text{inner}} = A + (0 - A)e^{-t/\varepsilon}$ where

$$A = \exp\left(\int_0^1 \frac{d\tau}{\tau + 1}\right) = 2.$$

So $x_{\text{inner}}(t) = 2 - 2e^{-t/\varepsilon}$.
Zeroth order approximation for $t \in [0, 1]$ is

$$x_a(t) = x_{\text{inner}}(t) + x_{\text{outer}} - A = \frac{2}{t + 1} - 2e^{-t/\varepsilon}.$$

b)

$$\varepsilon\ddot{x} + (\cosh t)\dot{x} - x = 0,\; x(0) = 1,\; x(1) = 1.$$

Comparison with theorem gives $p(t) = \cosh t$ and $q(t) = -1$ so p and q are cntinous on $[0, 1]$ with $p(t) > 0$ for $t \in [0, 1]$ as required.
The theorem gives

$$x_{\text{outer}} = \exp\left(\int_t^1 -\frac{d\tau}{\cosh\tau}\right) = \exp\left(-\int_t^1 \text{sech}\tau d\tau\right).$$

Since $-\int_t^1 \text{sech}\tau d\tau = -[\arctan(\sinh\tau)]_t^1 = \arctan(\sinh t) - \arctan(\sinh 1)$ so

$$x_{\text{outer}} = \exp\{\arctan(\sinh t) - \arctan(\sinh 1)\}.$$

Also the theorem gives $x_{\text{inner}} = A + (0 - A)e^{-t/\varepsilon}$ where

$$A = \exp\left(-\int_0^1 \frac{d\tau}{\cosh\tau}\right) = \exp\{-\arctan(\sinh 1)\}.$$

So the zeroth order approximation for $t \in [0, 1]$ is

$$\begin{aligned} x_a(t) &= x_{\text{inner}}(t) + x_{\text{outer}} - A \\ &= \exp\{\arctan(\sinh t) - \arctan(\sinh 1)\} + (1 - \exp\{-\arctan(\sinh 1)\})e^{-t/\varepsilon} \\ &= e^{-t/\varepsilon} + e^{-\arctan(\sinh 1)}(e^{\arctan(\sinh t)} - e^{-t/\varepsilon}). \end{aligned}$$