Methods of Applied Mathematics

Sheet 3 solutions

1. (a)

$$\ddot{x} + (1+\varepsilon)x = 0,\ x(0) = 1,\ \dot{x}(0) = 0.$$

As usual, put

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

to get

$$\begin{cases} (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \cdots)(1+\varepsilon)(x_0 + \varepsilon x_1 + \cdots) = 0, \\ x_0(0) + \varepsilon x_1(0) + \cdots = 1,\ \dot{x}_0(0) + \varepsilon \dot{x}_1(0) + \cdots = 0. \end{cases}$$

zeroth-order approximation:

$$\ddot{x}_0 + x_0 = 0,\ x_0(0) = 1,\ \dot{x}_0(0) = 0 \Rightarrow x_0 = \cos t$$

first-order correction:

$$\ddot{x}_1 + x_1 + x_0 = 0,\ x_1(0) = 0,\ \dot{x}_1(0) = 0 \Rightarrow \ddot{x}_1 + x_1 = -\cos t.$$

For PI try $x_p = At\cos t + Bt\sin t$. So $\dot{x}_p = A\cos t + B\sin t + t(-A\sin t + B\cos t)$ and $\ddot{x}_p = 2(-A\sin t + B\cos t) + t(-A\cos t - B\sin t)$.
Substitution gives

$$2(-A\sin t + B\cos t) + t(-A\cos t - B\sin t) + At\cos t + Bt\sin t = -\cos t$$

$$\Rightarrow -A2 = 0,\ 2B = -1 \quad \Rightarrow A = 0,\ B = -\frac{1}{2} \quad \text{for a solution.}$$

Hence $x_p = -\frac{1}{2}t\sin t$ and the general solution is

$$x_1 = C\cos t + D\sin t - \frac{1}{2}t\sin t.$$

So $\dot{x}_1 = C\sin t + D\cos t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t$. Initial conditions give $C = 0,\ D = 0$ so

$$x_1 = -\frac{1}{2}t\sin t$$

which has a secular term.
to remove it, scale t to s using

$$s = \omega t,\ \omega = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots.$$

The BVP becomes

$$\omega^2 x'' + (1+\varepsilon)x = 0,\ x(0) = 1,\ x'(0) = 0$$

and

$$(1 + \omega_1\varepsilon + \cdots)^2(x_0'' + \varepsilon x_1'' + \cdots) + (1+\varepsilon)(x_0 + \varepsilon x_1 + \cdots) = 0$$
$$x_0(0) + \varepsilon x_1(0) + \cdots = 1,\ x_0'(0) + \varepsilon x_1'(0) + \cdots = 0.$$

Using terms in ε^0:

$$x_0'' + x_0 = 0,\ x_0(0) = 1,\ x_0'(0) = 0 \Rightarrow x_0 = \cos s.$$

Using terms in ε^1:

$$2\omega_1 x_0'' + x_1'' + x_1 + x_0 = 0,\ x_1(0) = 0,\ x_1'(0) = 0$$

$$\Rightarrow x_1'' + x_1 = -2\omega_1 x_0'' - x_0 = (2\omega_1 - 1)\cos s,\ x_1(0) = 0,\ x_1'(0) = 0.$$

So choosing $\omega_1 = \frac{1}{2}$ gets rid of the secular term.

The BVP for x_1 becomes

$$x_1'' + x_1 = 0,\ x_1(0) = 0,\ x_1'(0) = 0 \Rightarrow x_1 = 0.$$

So the solution valid up to first order is

$$x_a(t) = \cos(1 + \frac{1}{2}\varepsilon)t.$$

Approximate period τ_a is given by

$$(1 + \frac{1}{2}\varepsilon)\tau_a = 2\pi \Rightarrow \tau_a = 2\pi(1 + \frac{1}{2}\varepsilon)^{-1}$$

$$\Rightarrow \tau_a = 2\pi(1 - \frac{1}{2}\varepsilon) \text{ to first order.}$$

(b) Exact solution is

$$x = \cos((1 + \varepsilon)^{1/2}t)$$

which gives an exact period of

$$\tau = 2\pi(1 + \varepsilon)^{-1/2} = 2\pi(1 - \frac{1}{2}\varepsilon + \cdots)$$

which agrees with τ_a up to first order in ε.

2. Dimensionless variables are:

$$\phi = \theta/\theta_c = \theta/\varepsilon, \quad t = \tau/\tau_c = \tau/\sqrt{a/g} \quad \Rightarrow \theta = \varepsilon\phi,\ \tau = t\sqrt{a/g}.$$

Equation of motion becomes

$$\frac{\varepsilon}{(a/g)}\frac{d^2\phi}{dt^2} + \frac{g}{a}\sin(\varepsilon\phi) = 0$$

$$\Rightarrow \frac{d^2\phi}{dt^2} + \frac{\sin(\varepsilon\phi)}{\varepsilon} = 0.$$

Since

$$\frac{d\theta}{d\tau} = \frac{\varepsilon}{\sqrt{a/g}}\frac{d\phi}{dt}$$

the initial condition becomes $\frac{d\phi}{dt} = 0$ and the initial condition $\theta = \varepsilon$ becomes $\phi = 1$.
For perturbation we need to expand:

$$\frac{\sin(\varepsilon\phi)}{\varepsilon} = \frac{1}{\varepsilon}(\varepsilon\phi - \frac{1}{3!}\varepsilon^2\phi^3 + \cdots) = \phi - \frac{1}{6}\varepsilon^2\phi^3 = \cdots$$

Then trying $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$ gives

$$(\ddot{\phi}_0 + \varepsilon\ddot{\phi}_1 + \varepsilon^2\ddot{\phi}_2 + \cdots) + (\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots) - \frac{1}{6}\varepsilon^2(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots)^3 = 0,$$

with

$$\dot{\phi}_0 + \varepsilon\dot{\phi}_1 + \varepsilon^2\dot{\phi}_2 + \cdots = 0, \quad \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots = 1.$$

Zeroth order approximation is given by coefficients of ε^0:

$$\ddot{\phi}_0 + \phi_0 = 0,\ \dot{\phi}_0(0) = 0,\ \phi_0(0) = 1$$

$$\Rightarrow \phi_0 = A\cos t + B\sin t,\ \dot{\phi}_0 = -A\sin t + B\cos t$$

with $B = 0,\ A = 1$. So $\phi_0 = \cos t$.
First order correction is given by coefficients of ε^1:

$$\ddot{\phi}_1 + \phi_1 = 0,\ \dot{\phi}_1(0) = 0,\ \phi_1(0) = 0 \quad \Rightarrow \phi_1 = 0.$$

Second order correction is given by coefficients of ε^2:

$$\ddot{\phi}_2 + \phi_2 - \frac{1}{6}\phi_0^3 = 0,\ \dot{\phi}_2(0) = 0,\ \phi_2(0) = 0 \quad \Rightarrow \ddot{\phi}_2 + \phi_2 = \frac{1}{6}\cos^3 t = \frac{1}{24}(\cos 3t + 3\cos t).$$

Solving $\ddot{\phi}_2 + \phi_2 = 0$ gives $\phi_c = A\cos t + B\sin t$, for a particular solution we try

$$\phi_p = C\cos 3t + Dt\sin t + Et\cos t.$$

Substituting into the differential equations gives

$$-9C\cos 3t + 2D\cos t - Dt\sin t - 2E\sin t - Et\cos t + C\cos 3t + Dt\sin t + Et\cos t = \frac{1}{24}\cos 3t + \frac{1}{8}\cos t.$$

$$\Rightarrow \ -8C = \frac{1}{24},\ 2D = \frac{1}{8},\ -2E = 0,\ \Rightarrow\ C = -\frac{1}{192},\ D = \frac{1}{16},\ E = 0.$$

So $\phi_2 = -\frac{1}{192}\cos 3t + \frac{1}{16}t\sin t + A\cos t + B\sin t$. Using the initial conditions gives $A = \frac{1}{192}$ and $B = 0$. So

$$\phi_2 = \frac{1}{192}(\cos t - \cos 3t) + \frac{1}{16}t\sin t.$$

Hence the solution contains the secular term $\frac{\varepsilon^2}{16}t\sin t$ that is present in ϕ_2.

3. Scaled problem (that gives secular term) is

$$\frac{d^2\phi}{dt^2} + \{\phi - \frac{1}{6}\varepsilon^2\phi^3 + \cdots\} = 0,\ \dot{\phi}(0) = 0,\ \phi(0) = 1.$$

Introduce new timescale $s = \omega t$ where

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2$$

Problems becomes

$$\omega^2\phi'' + \{\phi - \frac{1}{6}\varepsilon^2\phi^3 + \cdots\} = 0,\ \phi'(0) = 0,\ \phi(0) = 1,$$

where $'$ denotes differentaition with respect to s. So

$$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2)^2(\phi_0'' + \varepsilon\phi_1'' + \varepsilon^2\phi_2'') + (\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots) - \frac{\varepsilon^2}{6}(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots)^3 + \cdots = 0$$

with

$$\phi_0'(0) + \varepsilon\phi_1'(0) + \varepsilon^2\phi_2' + \cdots = 0,\quad \phi_0(0) + \varepsilon\phi_1(0) + \varepsilon^2\phi_2 + \cdots = 1.$$

Coefficients of ε^0:

$$\phi_0'' + \phi_0 = 0,\ \phi_0'(0) = 0,\ \phi_0(0) = 1\ \Rightarrow\ \phi_0 = \cos s \ \text{ as before.}$$

Coefficients of ε^1:

$$\phi_1'' + 2\omega_1\phi_0'' + \phi_1 = 0,\ \phi_1'(0) = 0,\ \phi_1(0) = 0.$$

To solve $\phi_1'' + \phi_1 = -2\omega_1\phi_0'' = 2\omega_1\cos s$ we first solve the homogenous equation $\phi_1'' + \phi_1 = 0$ to obtain

$$\phi_1 = A\cos s + B\sin s.$$

For a particular solution we try

$$\phi_p = s(C\cos s + D\sin s)$$

but this will give a secular term so we need to remove the $\cos s$ term from the right hand side of the equation. I.e. we should take

$$\omega_1 = 0.$$

Hence $\phi_1 = A\cos s + B\sin s$ and using the boundary conditions $\phi_1'(0) = 0$ and $\phi_1(0) = 0$ gives $\phi_1 = 0$ as before.

Coefficients of ε^2:

$$\phi_2'' + 2\omega_1\phi_1'' + (\omega_1^2 + 2\omega_2)\phi_0'' = \phi_2 - \frac{1}{6}\phi_0^3 = 0,\ \phi_2'(0) = 0,\ \phi_2(0) = 0.$$

Since we took $\omega_1 = 0$, this reduces to

$$\phi_2'' + 2\omega_2\phi_0'' + \phi_2 = \frac{1}{6}\phi_0^3,\ \phi_2'(0) = 0,\ \phi_2(0) = 0.$$

Putting $\phi_0 = \cos s$ and noting that $\cos^3 s = \frac{1}{4}(\cos 3s + 3\cos s)$ gives

$$\phi_2'' - 2\omega_2\cos s + \phi_2 = \frac{1}{6}\phi_0^3 = \frac{1}{24}(\cos 3s + \cos s),\ \phi_2'(0) = 0,\ \phi_2(0) = 0.$$

$$\Rightarrow\ \phi_2'' + \phi_2 = \frac{1}{24}\cos 3s + \cos s(2\omega_2 + \frac{3}{24}),\ \phi_2'(0) = 0,\ \phi_2(0) = 0.$$

So the secular term is removed by taking $-2\omega_2 = \frac{3}{24} \ \Rightarrow\ \omega_2 = -\frac{1}{16}$. To second order we get $\omega = 1 - \frac{1}{16}\varepsilon^2$. Correction period is given by $s = 2\pi$

$$\Rightarrow \omega t = 2\pi\ \Rightarrow\ t = 2\pi/\omega = 2\pi(1 - \frac{1}{16}\varepsilon^2)^{-1}\ \Rightarrow\ t = 2\pi(1 + \frac{1}{16}\varepsilon^2) \text{ to second order}$$

hence $\tau = 2\pi(1 + \frac{\varepsilon^2}{16})\sqrt{a/g}$.

4. Suppose $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, substituting into $x^3 - (4+\varepsilon)x + 2\varepsilon = 0$ gives

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 - (4+\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 2\varepsilon = 0$$

Coefficient of ε^0:

$$x_0^3 - 4x_0 = 0\ \Rightarrow\ x_0(x_0^2 - 4) = 0\ \Rightarrow\ x_0 = 0\ \text{ or }\ x_0 = \pm 2.$$

Coefficient of ε^1:

$$3x_0^2 x_1 - 4x_1 - x_0 + 2 = 0\ \Rightarrow\ (3x_0^2 - 4)x_1 = x_0 - 2\ \Rightarrow\ x_1 = \frac{x_0 - 2}{3x_0^2 - 4}.$$

Coefficient of ε^2:

$$3x_0 x_1^2 + 3x_0^2 x_2 - 4x_2 - x_1 = 0\ \Rightarrow\ (3x_0^2 - 4)x_2 = x_1 - 3x_0 x_1^2\ \Rightarrow\ x_2 = \frac{x_1 - 3x_0 x_1^2}{3x_0^2 - 4}.$$

Since $\varepsilon = 0.001$ for the given equation, for 6 decimal places accuracy we need to go no further than ε^2.

<u>Three roots</u>

(a)

$$x_0 = 0 \Rightarrow x_1 = \frac{-2}{-4} = 0.5\ \Rightarrow x_2 = \frac{0.5}{-4} = -0.125.$$

So root is

$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 = 0 + 0.0005 - (0.000001 \times 0.125) = 0.000500 (\text{to 6 dec.pl.}).$$

(b)

$$x_0 = 2 \Rightarrow x_1 = 0\ \Rightarrow x_2 = 0.$$

So root is 2.000000 to 6 dec. pl. (In fact 2 is an exact root.)

(c)

$$x_0 = -2 \Rightarrow x_1 = \frac{-4}{12 - 4} = -0.5\ \Rightarrow x_2 = \frac{-0.5 + 6(0.5)^2}{3(2)^2 - 4} = 0.125.$$

So root is

$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 = -2 - 0.0005 + (0.000001 \times 0.125) = -2.000500 (\text{to 6 dec.pl.}).$$