Methods of Applied Mathematics

Sheet 2 solutions

1. Problem in original variables:

$$\frac{d^2h}{dt^2} = -\frac{R^2g}{(R+h)^2}, \; h(0) = 0, \; \frac{dh}{dt}(0) = V.$$

Version A

$$\begin{cases} h_c = R \\ t_c = R/V \end{cases} \Rightarrow \begin{cases} \bar{h} = h/R \\ \bar{t} = Vt/R \end{cases} \Rightarrow \begin{cases} h = R\bar{h} \\ t = R\bar{t}/V. \end{cases}$$

So the equation becomes

$$\frac{d^2R\bar{h}}{d(R\bar{t}/V)^2} = -\frac{R^2g}{(R+R\bar{h})^2} \Rightarrow \frac{V}{R}\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{g}{(1+\bar{h})^2}$$

$$\Rightarrow \varepsilon\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2} \quad \text{where } \varepsilon = V^2/gR.$$

Clearly $h(0) = 0, \; \bar{h}(0) = 0$ and

$$\frac{dh}{dt}(0) = V \Rightarrow \frac{d(R\bar{h})}{d(R\bar{t}/V)}(0) = V \Rightarrow \frac{d\bar{h}}{d\bar{t}}(0) = 1.$$

Version B

$$\begin{cases} h_c = R \\ t_c = \sqrt{R/g} \end{cases} \Rightarrow \begin{cases} h = R\bar{h} \\ t = \sqrt{R/g}\bar{t}. \end{cases}$$

So the equation becomes

$$\frac{d^2R\bar{h}}{d(\sqrt{R/g}\bar{t})^2} = -\frac{R^2g}{(R+R\bar{h})^2} \Rightarrow g\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{g}{(1+\bar{h})^2}$$

$$\Rightarrow \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2} \quad \text{where } \varepsilon = V^2/gR.$$

Clearly $h(0) = 0, \; \bar{h}(0) = 0$ and

$$\frac{dh}{dt}(0) = V \Rightarrow \frac{d(R\bar{h})}{d(\sqrt{R/g}\bar{t})}(0) = V \Rightarrow \frac{d\bar{h}}{d\bar{t}}(0) = \sqrt{\varepsilon}.$$

Version C

$$\begin{cases} h_c = V^2/g \\ t_c = V/g \end{cases} \Rightarrow \begin{cases} h = V^2\bar{h}/g \\ t = V\bar{t}/g. \end{cases}$$

So the equation becomes

$$\frac{d^2V^2\bar{h}/g}{d(V\bar{t}/g)^2} = -\frac{R^2g}{(R+V^2\bar{h}/g)^2} \Rightarrow g\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{g}{(1+V^2\bar{h}/Rg)^2}$$

$$\Rightarrow \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\varepsilon\bar{h})^2} \quad \text{where } \varepsilon = V^2/Rg.$$

Clearly $h(0) = 0, \; \bar{h}(0) = 0$ and

$$\frac{dh}{dt}(0) = V \Rightarrow \frac{d(V^2\bar{h}/g)}{d(V\bar{t}/g)}(0) = V \Rightarrow \frac{d\bar{h}}{d\bar{t}}(0) = 1.$$

2. V small $\Rightarrow h'$ small (while travelling upwards), so 2^{nd} term of $h''+kh'+g=0$ is small and equation of motion is approximated by $h''+g=0$ (amounting to neglecting air-resistance). The quantities V^2/g and V/g are twice the height attained and the time taken to achieve max height, in this approximation. Since the actual values are close to these approximations, we are justified in using them as characteristic values.

   Using $h_c = V^2/g$ and $t_c = V/g$ to scale to dimensionless variables

$$\bar{h} = h/h_c, \quad \bar{t} = t/t_c,$$

   we get

$$\frac{d^2\bar{h}}{d\bar{t}^2} + \frac{kV}{g}\frac{d\bar{h}}{d\bar{t}} + 1 = 0$$

$$\bar{h}(0) = 0, \ \frac{d\bar{h}}{d\bar{t}}(0) = 1$$

   for the rescaled problem. Neglecting the middle term amounts to regarding $\varepsilon = kV/g$ as small, so V is small with respect to g/k.

   If V is large then (initially) h' is large and the resisitive-force term kh' will be large compared to the gravitational-force term g. So initial motion can be approximated by

$$h'' + kh' = 0 \qquad \text{or} \qquad u' + ku = 0$$

   where $u' = h$=upward velocity. This first-order equation for u has solution $u = ve^{-kt}$ (on using $u(0) = v$). Noting that u decayes exponentially and that $\frac{1}{k}\ln 2$ is its half-life suggests using $t_c = \frac{1}{k}$ as a characteristic time. To obtain a characteristic value for h we integrate u with respect to t and use the initial data $h(0) = 0$ to obtain $h = \frac{V}{k}(1 - e^{-kt})$. We take h_c to be the maximum value of h i.e. take $h_c = V/k$.

3. **a** $u = f(t) = A\cos(\lambda t), \ t \in [0, \infty), \ A, \lambda > 0.$

$$u_c = \sup_{t\in[0,\infty)} |f(t)| = \sup_{t\in[0,\infty)} |A\cos(\lambda t)| = A.$$

   As

$$\sup_{t\in[0,\infty)} |f'(t)| = \lambda A$$

   we take $t_c = u_c/\lambda A = \frac{1}{\lambda}$.

   b $u = f(t) = \exp(-at), \ t \in [0, \infty), \ a > 0.$

$$u_c = \sup_{t\in[0,\infty)} |f(t)| = 1 \quad (\text{as } f(t) \text{ decreases with } t \text{ and } f(0) = 1).$$

   As

$$\sup_{t\in[0,\infty)} |f'(t)| = a$$

   we take $t_c = u_c/a = \frac{1}{a}$.

   c $u = f(t) = 100\exp\left(\frac{1-t}{1000}\right), \ t \in [0, 1].$

$$\sup_{t\in[0,1]} |f(t)| = 100\exp\left(\frac{1-t}{1000}\right) \approx 100$$

   so take $u_c = 100$ (which is more inconvenient). As

$$|f'(t)| = \left|-\frac{1}{10}\exp\left(\frac{1-t}{1000}\right)\right| = \frac{1}{10}\exp\left(\frac{1-t}{1000}\right) \approx \frac{1}{10}.$$

   So we take $t_c = u_c/(1/10) = 10u_c = 1000.$