Methods of Applied Mathematics

Sheet 1 solutions

1. The law is of the form $f(v, r, \rho, g, \mu, \sigma) = 0$, where

$$f(v, r, \rho, g, \mu, \sigma) = v - \frac{2r^2\rho g}{9\mu}(1 - \frac{\sigma}{\rho}).$$

Now $[v] = LT^{-1}$, $[r] = L$, $[\rho] = ML^{-3}$, $[g] = LT^{-2}$, $[\mu] = ML^{-1}T^{-1}$ and $[\sigma] = ML^{-3}$, so changes in units of $M,\ L,\ T$, giving rise to changes $\bar{m} = \alpha m$, $\bar{l} = \beta l$, $\bar{t} = \gamma t$ in the corresponding quantities, induce the changes

$$\bar{v} = \beta\gamma^{-1}v,\ \bar{r} = \beta r,\ \bar{\rho} = \alpha\beta^{-3}\rho,\ \bar{g} = \beta\gamma^{-2}g,\ \bar{\mu} = \alpha\beta^{-1}\gamma^{-1}\mu,\ \bar{\sigma} = \alpha\beta^{-3}\sigma.$$

Then

$$\begin{aligned} f(\bar{v}, \bar{r}, \bar{\rho}, \bar{g}, \bar{\mu}, \bar{\sigma}) &= \bar{v} - \frac{2\bar{r}^2\bar{\rho}\bar{g}}{9\bar{\mu}}(1 - \frac{\bar{\sigma}}{\bar{\rho}}) \\ &= (\beta\gamma^{-1}v) - \frac{2(\beta r)^2(\alpha\beta^{-3}\mu)(\beta\gamma^{-2}g)}{9(\alpha\beta^{-1}\gamma^{-1}\mu)}(1 - \frac{\sigma}{\rho}) \\ &= \beta\gamma^{-1}\left\{v - \frac{2r^2\rho g}{9\mu}(1 - \frac{\sigma}{\rho})\right\} \\ &= \beta\gamma^{-1}f(v, r, \rho, g, \mu, \sigma). \end{aligned}$$

So $f(\bar{v}, \bar{r}, \bar{\rho}, \bar{g}, \bar{\mu}, \bar{\sigma}) = 0 \Leftrightarrow f(v, r, \rho, g, \mu, \sigma) = 0$ showing that the law is unit free.

2. The dimension matrix for the law is

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} m & l & t & P & \rho \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & -2 & 0 \end{pmatrix} \end{array} = A$$

Since A is in Echelon form we have that rank $A = 3$ and hence there are 2 independent solutions (dimension of the solution space is 2). Clearly $[0, 3, -1, 0, 1]^T$ and $[3, 0, -2, 6, 1]^T$ solve $A\underline{x} = \underline{0}$ and since the (5×2) matrix

$$B = \begin{pmatrix} 0 & 3 \\ 3 & 0 \\ -1 & -2 \\ 0 & 6 \\ 1 & 1 \end{pmatrix}$$

can be transformed using row operations into one which contains the (2×2) identity matrix. The two solutions yield the dimesnionless quantities

$$\pi_1 = \frac{l^3\rho}{m} \quad \text{and} \quad \pi_2 = \frac{P^3t^6}{m^2\rho}$$

which implies by the Buckingham Pi theorem an equivalent law of the form

$$G(\frac{l^3\rho}{m}, \frac{P^3t^6}{m^2\rho}) = 0.$$

3. The first assumption says that there is a law of the form $f(A, c, \phi) = 0$ for which the dimension matrix is

$$\begin{array}{cc} & \begin{array}{ccc} A & c & \phi \end{array} \\ L & \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} = A \end{array}$$

Since rank $A = 1$ the solution space has dimension 2 and a basis is $[-\frac{1}{2},\ 1,\ 0]^T$, $[0,\ 0,\ 1]^T$ or

$$\pi_1 = \frac{c}{A^{1/2}}, \quad \pi_2 = \phi$$

and hence there is an equivalent law of the form

$$G(\frac{c}{A^{1/2}}, \phi) = 0$$

which may be rewritten as $\frac{c}{A^{1/2}} = h(\phi)$ or

$$A = c^2 g(\phi).$$

Since $A = A_1 + A_2$ and ϕ is the smallest angle in all of the triangles by similarity we find

$$c^2 g(\phi) = a^2 g(\phi) + b^2 g(\phi)$$
$$\Rightarrow c^2 = a^2 + b^2$$

as required for Pythagarus's Theorem.