Methods of Applied Mathematics

Sheet 5. Calculus of Variations

In the following exercises, the instruction to find possible extremals should be interpreted as an instruction to find solutions of the appropriate Euler equation. That is, to find *critical functions* or *stationary functions*, which are candidates for extremals.

1. Show that straightforward applications of the Euler equation fail to find the extremals $y \in C^2[0,1]$ of the following functionals:

   (a)
   $$J(y) = \int_0^1 y'\,dx, \quad y(0) = 0, \quad y(1) = 1$$

   (b)
   $$J(y) = \int_0^1 yy'\,dx, \quad y(0) = 0, \quad y(1) = 1$$

   (c)
   $$J(y) = \int_0^1 xyy'\,dx, \quad y(0) = 0, \quad y(1) = 1.$$

2. Find possible extremals of the functional
   $$J(y) = \int_a^b (x^2 y'^2 + y^2)\,dx, \qquad y \in C^2[a,b],$$
   that satisfy fixed-endpoint conditions at a and b.

3. A theorem given in lectures gave a necessary condition for an extremal of a functional involving a Lagrangian in which the lefthand endpoint was fixed, but the righthand endpoint was free. Extend this theorem to the case where both endpoints are free and give a proof of it.

4. Find possible extremals $y \in C^2[0,1]$ of the following functionals:

   (a)
   $$J(y) = \int_0^1 (y'^2 + y^2)\,dx, \quad y(0) = 1, \quad y(1) \text{ unspecified}$$

   (b)
   $$J(y) = \int_0^1 (y'^2 + 2y'y + 2y' + y)\,dx, \quad y(0) = 2, \quad y(1) \text{ unspecified}$$

5. Find a possible extremal $y \in C^4[0,1]$ of the following functional:
   $$J(y) = \int_0^1 (1 + y''^2)\,dx,$$
   that satisfies $y(0) = 0$, $y'(0) = 0$, $y(1) = 1$, $y'(1) = 1$.