Methods of Applied Mathematics

Sheet 4. Singular Perturbations

1. Consider the cubic equation

$$\varepsilon x^3 + x - 2 = 0, \qquad 0 < \varepsilon \ll 1.$$

Use the direct method of perturbation to find the root close to 2, working as far as the first-order correction.
Use a balancing argument in order to rescale x, so that a perturbation using the rescaled variable leads to the other two roots. By taking a perturbation series in powers of $\sqrt{\varepsilon}$, find these roots, again working as far as the first-order correction.
Check your working by computing the product of the three roots found.

2. Use singular perturbation methods to obtain approximations, valid for $t \in [0, 1]$, to the following boundary-value problems in which $0 < \varepsilon \ll 1$.

(a)
$$\varepsilon \ddot{x} + 2\dot{x} + e^x = 0, \quad x(0) = 0, \quad x(1) = 0$$

(b)
$$\varepsilon \ddot{x} + \dot{x} = 2(1 - t), \quad x(0) = 1, \quad x(1) = 1.$$

3. Try doing the following problem by the method given in lectures:

$$\varepsilon \ddot{x} - \dot{x} = 2t, \quad x(0) = 1, \quad x(1) = 1, \qquad (0 < \varepsilon \ll 1).$$

You should find that the matching stage cannot be completed.
What is going wrong? How might you put it right?

4. Use Theorem 2.11 to obtain approximations, valid for $t \in [0, 1]$, to the following boundary-value problems in which $0 < \varepsilon \ll 1$.

(a)
$$\varepsilon \ddot{x} + (t + 1)\dot{x} + x = 0, \quad x(0) = 0, \; x(1) = 1$$

(b)
$$\varepsilon \ddot{x} + (\cosh t)\dot{x} - x = 0, \quad x(0) = 1, \quad x(1) = 1.$$

Hint:
$$\int_a^b \text{sech } t dt = [\arctan(\sinh t)]_a^b$$