Methods of Applied Mathematics

Sheet 3. Regular Perturbation

1. Consider the initial-value problem:-

$$\ddot{x} + (1+\epsilon)x = 0,$$
$$t \in (0,\infty), \quad 0 < \epsilon << 1, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

   (a) Show that if a solution is sought by the direct method of perturbation using $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ then the first-order correction x_1 contains a secular term. Use the Poincaré method to remove this secular term and hence obtain a solution valid up to terms of first-order in ϵ.

   What, to first-order, is the period of the solution?

   (b) Solve the problem exactly, expand your solution in powers of ϵ, and hence check your answer to part (a).

2. The equation of motion for a simple pendulum of length a is

$$\frac{d^2\theta}{d\tau^2} + \frac{g}{a}\sin\theta = 0,$$

   where θ is the angular displacement from the vertical.
   Suppose that the pendulum is released from rest with ϵ as the initial value of θ and that we take $\theta_c = \epsilon, \quad \tau_c = \sqrt{a/g}$ as characteristic values for θ and τ to arrive at new, scaled, dimensionless variables

$$\phi = \theta/\theta_c, \quad t = \tau/\tau_c.$$

   Show that, in terms of the dimensionless scaled variables, the equation of motion becomes

$$\frac{d^2\phi}{dt^2} + \frac{\sin\epsilon\phi}{\epsilon} = 0,$$

   with $d\phi/dt = 0$ and $\phi = 1$ at $t = 0$ as initial conditions.
   Use the direct method of perturbation to develop a solution to this problem up to terms quadratic in ϵ and show that this gives a secular term.
   Hint: Recall that

$$\frac{\sin(\varepsilon\phi)}{\varepsilon} = \frac{1}{\varepsilon}\left(\varepsilon\phi - \frac{1}{3!}\varepsilon^3\phi^3 + \cdots\right)$$

3. Rework Ex 2 using the Poincaré method to get rid of the secular term and hence find the second-order approximation to the period of the pendulum.

4. By applying the direct method of perturbation to the polynomial equation

$$x^3 - (4+\epsilon)x + 2\epsilon = 0$$

   obtain the three roots of the cubic equation

$$x^3 - 4.001x + 0.002 = 0,$$

   correct to six decimal places.