3 Singular perturbation theory

3.1 Big 'O' and Twiddles

In connection with approximations and their validity, it is useful to have notation that allows us to compare functions as some limit is approached.

Definition 3.1. *We say that* $f(x) = O(g(x))$ *as* $x \to x_0$ *if there exists* $\eta > 0$ *and an* $M > 0$ *such that*

$$|x - x_0| < \eta \quad \Rightarrow |f(x)| \leq M|g(x)|.$$

That is, there exists a neighbourhood of x_0 *and a positive constant* M *such that for all* x *in the neighbourhood, we have*

$$|f(x)| \leq M|g(x)|.$$

Definition above deals with a finite x_0. For infinite x_0, we have

Definition 3.2. *We say that* $f(x) = O(g(x))$ *as* $x \to \infty$ *if there exists* $X > 0$ *and an* $M > 0$ *such that*

$$x > X \quad \Rightarrow |f(x)| \leq M|g(x)|.$$

Remark 3.1. *We note that* $f(x) = O(1)$ *as* $x \to x_0$*, means that* x_0 *has a neighbourhood in which* $f(x)$ *is bounded.*

Example 3.1. *We have* $\cos x - 1 = O(x^2)$ *as* $x \to 0$.

Solution:
A truncation of Maclaurin's series gives

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(\eta_x), \; 0 < \eta_x < x,$$

where η_x depends on x.
Setting

$$f(x) = \cos x \quad \Rightarrow \qquad f(0) = 1, \quad f'(0) = 0 \text{ and } f''(\eta_x) = -\cos\eta_x.$$

Hence,

$$\cos x - 1 = -\frac{x^2\cos\eta_x}{2} \quad \Rightarrow \quad |\cos x - 1| \leq \frac{x^2}{2}|\cos\eta_x| \leq \frac{x^2}{2} \Rightarrow \quad \cos x - 1 = O(x^2) \qquad \forall\, x \in \mathbb{R}.$$

Definition 3.3. *We say that* $f(x) \sim g(x)$ *as* $x \to x_0$ *(*$f(x)$ *'twiddles'* $g(x)$*) provided*

$$\frac{f(x)}{g(x)} \to K, \;\; \text{as } x \to x_0,$$

where $K \neq 0$ *and* $K \neq \infty$.

$f(x) \sim g(x)$ is equivalent to $f(x) = O(g(x))$ and $g(x) = O(f(x))$ as $x \to x_0$. We can use this definition for $x_0 = \infty$.
We shall use 'twiddles' in comparing objects like $f(\varepsilon)$ and $g(\varepsilon)$ as $\varepsilon \to 0$.

3.2 An algebraic example using singular perturbation theory.

Example 3.2. *Find the roots of*

$$\varepsilon x^2 + 2x + 1 = 0, \tag{3.1}$$

where $0 < \varepsilon \ll 1$*, by a perturbation method.*

Solution:
We assume $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, and substitute

$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 + 2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 1 = 0.$$

Equating coefficients of powers of ε^k on left and right gives

$$2x_0 + 1 = 0, \tag{3.2}$$
$$x_0^2 + 2x_1 = 0, \tag{3.3}$$
$$2x_0 x_1 + 2x_2 = 0. \tag{3.4}$$

Solving (3.2)-(3.4) gives

$$x_0 = -\frac{1}{2}, \quad x_1 = -\frac{1}{8} \quad \text{and} \quad x_2 = -\frac{1}{16}.$$

So we generate

$$x = -\frac{1}{2} - \frac{1}{8}\varepsilon - \frac{1}{16}\varepsilon^2 - \cdots,$$

which is the power series expansion in ε^k for the root closest to $-1/2$.
The method fails to give a formula for the second root which we know (because the product of the roots is $1/\varepsilon$).
The problem is *singular* because setting $\varepsilon = 0$ (equivalent to looking at zero order approximation) gives $2x + 1 = 0$ which has a lower degree than the original equation. (Analogous to differential equation being of lower order.)
What is wrong?
Our method assures that the equation is correctly scaled and therefore that εx^2 is small when ε is small. This is true for the root close to "$-1/2$" but not true for the root close to $-2/\varepsilon$ where $\varepsilon x^2 \sim 4/\varepsilon$. In effect we have a 'multi scale' problem and to deal correctly with the other root we must rescale x.
Example 3.2 (Continued)
Rescale x by putting $\overline{x} = x/\delta$ where δ is to be determined in terms of ε. Equation becomes

$$\varepsilon(\delta\overline{x})^2 + 2(\delta\overline{x}) + 1 = 0,$$
$$\varepsilon\delta^2\overline{x}^2 + 2\delta\overline{x} + 1 = 0.$$

We require this to be 'correctly' scaled so that $\overline{x}$ is of 'moderate' size when close to the unknown 2nd root. Possible candidates for δ are obtained by a process known as *balancing the coefficients.*
There are 3 possible balances.

a) 1st and 2nd terms of same order

$$\varepsilon\delta^2 \sim 2\delta \;\Leftrightarrow\; \delta \sim \varepsilon^{-1}.$$

b) 1st and 3rd terms of same order

$$\varepsilon\delta^2 \sim 1 \;\Leftrightarrow\; \delta \sim \varepsilon^{-1/2}.$$

c) 2nd and 3rd terms of same order

$$2\delta \sim 1 \;\Leftrightarrow\; \delta \sim 1.$$

We choose one of these in order to find the second root.
c) Setting $\delta = 1$ does not rescale x so we reject this choice.
b) Setting $\delta = \varepsilon^{-1/2}$ gives

$$\overline{x}^2 + 2\varepsilon^{-1/2}\overline{x} + 1 = 0,$$
$$\Rightarrow \quad \varepsilon^{1/2}\overline{x}^2 + 2\overline{x} + \varepsilon^{1/2} = 0.$$

We take

$$\bar{x} = \bar{x_0} + \varepsilon^{1/2}\bar{x_1} + \varepsilon\bar{x_2} + \cdots.$$

and to obtain the zeroth order approximation we set $\varepsilon = 0$ to obtain

$$2\bar{x}_0 = 0 \quad \Rightarrow \bar{x}_0 = 0.$$

$$\Rightarrow \bar{x} = \varepsilon^{1/2}\bar{x}_1 + \cdots .$$

We rejected this because $\bar{x}$ is not of moderate size.
a) Setting $\delta = \varepsilon^{-1}$ gives

$$\bar{x}^2 + 2\bar{x} + \varepsilon = 0. \tag{3.5}$$

Takng

$$\bar{x} = \bar{x}_0 + \varepsilon\bar{x}_1 + \varepsilon^2\bar{x}_2 + \cdots$$

yields

$$(\bar{x}_0 + \varepsilon\bar{x}_1 + \varepsilon^2\bar{x}_2 + \cdots)^2 + 2(\bar{x}_0 + \varepsilon\bar{x}_1 + \varepsilon^2\bar{x}_2 + \cdots) + \varepsilon = 0$$

and equating coefficients of ε^k gives

$$(\varepsilon^0) \qquad \bar{x}^2 + 2\bar{x}_0 = 0, \tag{3.6}$$

$$(\varepsilon^1) \qquad 2\bar{x}_0\bar{x}_1 + 2\bar{x}_1 + 1 = 0, \tag{3.7}$$

$$(\varepsilon^2) \qquad 2\bar{x}_0\bar{x}_2 + \bar{x}_1^2 + 2\bar{x}_2 = 0, \tag{3.8}$$

etc . . . .
From (3.6) we have that $\bar{x}_0 = 0$ or $\bar{x}_0 = -2$.
Taking $\bar{x}_0 = -2$, we get $\bar{x}_1 = 1/2$ and $\bar{x}_2 = 1/8$.

$$\Rightarrow \bar{x} = -2 + \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots .$$

Hence rewriting In terms of the original variable, $x = \delta\bar{x} = \bar{x}/\varepsilon$ we have

$$x = -\frac{2}{\varepsilon} + \frac{1}{2} + \frac{1}{8}\varepsilon + \cdots ,$$

which is the missing large root.
Actually (although we are entitled to discard $\bar{x}_0 = 0$ because this would lead to $\bar{x} = O(\varepsilon)$ which is not of moderate size as we require $\bar{x} \sim 1$) working with $\bar{x}_0 = 0$ gives

$$x = -\frac{1}{2} - \frac{\varepsilon}{8} + \cdots .$$

3.3 An BVP example using singular perturbation theory

For BVPs, singular perturbation problems arise when the direct method results in a succession of BVPs that cannot be satisfied, typically because the DE for the zeroth order approximation involves an order less than that of the DE in the original problem. The consequence is that we have more BCs than are needed for the DE of lesser order which in general cannot be simultaneously satisfied.
This is what happens in the following example.

Example 3.3. *Use singular perturbation theory to find an approximate solution to the BVP*

$$\begin{aligned} &\varepsilon\ddot{x} + 2\dot{x} + x = 0, \qquad t \in (0,1), \\ &x(0) = 0,\ x(1) = 1, \quad 0 < \varepsilon \ll 1. \end{aligned} \tag{3.9}$$

Solution:
Apply the direct method: set

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

in (3.9) to obtain

$$\varepsilon(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots) + 2(\dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0$$

with

$$x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \cdots = 0,$$
$$x_0(1) + \varepsilon x_1(1) + \varepsilon^2 x_2(1) + \cdots = 1.$$

Equating coefficients of ε^0 gives :

$$2\dot{x}_0 + x_0 = 0,\ t \in (0,1), \quad x_0(0) = 0,\ x_0(1) = 1.$$

Solving $2\dot{x}_0 + x_0 = 0$ gives $x_0 = Ae^{-t/2}$ and to find A we use one of the two boundary conditions. We see that imposing $x_0(1) = 1$ gives $A = e^{1/2} \;\Rightarrow\; x_0(t) = e^{(1-t)/2}$, while imposing $x_0(0) = 0$ gives $A = 0 \;\Rightarrow\; x_0(t) = 0$.
We choose to use the right hand boundary condition $x_0(0) = 1$ and hence we have

$$x_0(t) = e^{(1-t)/2}.$$

Equating coefficients of ε^1 and using $\ddot{x}_0 = 1/4e^{(1-t)/2}$ gives :

$$\ddot{x}_0 + 2\dot{x}_1 + x_1 = 0,\ t \in (0,1), \quad x_1(0) = 0,\ x_1(1) = 0$$

$$\Rightarrow 2\dot{x}_1 + x_1 = -\frac{1}{4}e^{(1-t)/2},\ t \in (0,1), \quad x_1(0) = 0,\ x_1(1) = 0.$$

Again the ODE is first order and we obtain different solutions depending on which boundary condition we use. Using the right hand boundary condition $x_1(0) = 0$ we obtain a solution $x_1(t)$.
If we carry on in this way, ignoring the left hand boundary condition $x(0) = 0$, we can build up a solution that satisfies the differential equation and the boundary condition $x(1) = 1$ but not the boundary condition $x(0) = 0$.
On the other hand we could always use the boundary condition $x(0) = 0$ and ignore $x(1) = 1$.

A solution that satisfies the differential equation and the condition $x(1) = 1$ is called an *outer solution* as $x = 1$ can be thought of as the outer boundary of the domain $(0, 1)$. Its zeroth order approximation is

$$x_{\text{outer}}(t) = e^{(1-t)/2}$$

and we assume it is valid (in some sense) for t near 1. We are missing an inner solution which is valid for t near 0.
Let us assume that the difficulty is due to incorrect scaling for t near 0 (i.e., we have a multi-scale problem) and rescale by using $s = t/\delta$ as an independent variable, where δ is to be determined in terms of ε.
Let

$$\hat{x}(s) = x(t) \quad \Rightarrow \quad \frac{dx}{dt} = \frac{d\hat{x}}{ds}\frac{ds}{dt} = \frac{1}{\delta}\frac{d\hat{x}}{ds},$$

Rewriting the ODE in (3.9) in terms of $\hat{x}(s)$ gives

$$\frac{\varepsilon}{\delta^2}\frac{d^2\hat{x}}{ds^2} + \frac{2}{\delta}\frac{d\hat{x}}{ds} + \hat{x} = 0$$

and by convention we drop the '$\hat{\ }$' (abusing notation as we frequently do) to give

$$\frac{\varepsilon}{\delta^2}x'' + \frac{2}{\delta}x' + x = 0,\ x(0) = 0, \tag{3.10}$$

where $'$ denotes differential with respect to s.
There are three terms and three coefficients. We wish to choose δ so that we have correct scaling near $t = 0$ (or $s = 0$). We determine δ by *balancing*. That is, choose δ so that two of the coefficients of the ODE are of the same order (the 'dominant' balance) and the third one is small in comparison (and therefore missing from the zeroth order approximation) and hope for a solution that is valid near $t = 0$ ($s = 0$).

Possibilities

a) 2nd and 3rd terms same order

$$\Rightarrow\ 2/\delta \sim 1\ \Rightarrow\ \delta \sim 1\ \Rightarrow\ \text{original scaling and so reject.}$$

b) 1st and 3rd terms same order

$$\Rightarrow\ \frac{\varepsilon}{\delta^2} \sim 1\ \Rightarrow\ \delta \sim \sqrt{\varepsilon}.$$

Taking $\delta = \sqrt{\varepsilon}$ in (3.10) gives

$$x'' + \frac{2}{\sqrt{\varepsilon}}x' + x = 0$$
$$\Rightarrow\ \sqrt{\varepsilon}x'' + 2x' + \sqrt{\varepsilon}x = 0,$$

To obtain the zeroth order approximation we set $\varepsilon = 0$ to obtain the zeroth order approximation

$$2x_0' = 0 \quad \text{(Reduction of order)}$$
$$\Rightarrow\ x_0 = C$$

Applying the boundary condition $x(0) = 0$ gives $x_0 = 0$ and hence we obtain $x_0 = 0$ which is no good as $x = x_0 + \sqrt{\varepsilon}x_1 + \cdots = \sqrt{\varepsilon}x_1 + \cdots$ is not of moderate size so we reject this scaling.

c) 1st and 2nd terms same order

$$\Rightarrow\ \frac{\varepsilon}{\delta^2} \sim \frac{2}{\delta}\ \Rightarrow\ \delta \sim \varepsilon.$$

Taking $\delta = \varepsilon$ in (3.10) gives

$$x'' + 2x' + \varepsilon x = 0.$$

This looks promising as it is still a 2nd order ODE. Setting $\varepsilon = 0$ gives us that the zeroth order approximation x_0 satisfies

$$x_0'' + 2x_0' = 0\ \Rightarrow\ x_0 = Ae^0 + Be^{-2s} \qquad \text{(as roots of auxiliary equation are } -2, 0).$$

Applying the boundary condition at $s = 0\ \Rightarrow\ A = -B$ gives an approximate inner solution

$$x_0(s) = A(1 - e^{-2s}) \quad \text{or } x_0(t) = A(1 - e^{-2t/\varepsilon}).$$

So now we have an approximation outer solution (valid near $t = 1$)

$$x_{\text{outer}}(t) = \exp\left(\frac{1-t}{2}\right)$$

and approximation inner solution (valid near $t = 0$)

$$x_{\text{inner}}(t) = A\left(1 - \exp\left(-\frac{2t}{\varepsilon}\right)\right).$$

The final job to do is to choose A so that these solutions 'match' and hence obtain a composite solution valid everywhere in $(0, 1)$.

3.4 Matching

In the above example, we have an inner solution and an outer solution, each being valid for t in different parts of the interval. Roughly,

$$x_{\text{inner}} \text{ is valid where } t \sim \varepsilon \ \text{ and } \ x_{\text{outer}} \text{ is valid in the remaining interval.}$$

The idea of matching is to adjust the constant A in the inner solution so that in the overlap region the inner solution matches the outer solution (in a way to be explained).
We obtain the overlap (or intermediate) region by using an intermediate scale u which is the geometric

mean of t and s. This scale lies between the original scale t of the outer approximation and the scale $s = t/\varepsilon$ of the inner approximation and is given by $u = \sqrt{st}$. The condition for matching is then

$$\lim_{\varepsilon \searrow 0} x_{\text{outer}}\left(\sqrt{\varepsilon}u\right) = \lim_{\varepsilon \searrow 0} x_{\text{inner}}\left(\sqrt{\varepsilon}u\right) := CL \tag{3.11}$$

where CL denotes the common limit. The solutions must agree with each other in the limit $\varepsilon \searrow 0$ when written in terms of the intermediate scale u.

In our example we found that $s = t/\varepsilon$ is the right scaling for the inner solution. It follows that in this case $u = (t^2\varepsilon)^{1/2}$ or $u = t/\sqrt{\varepsilon}$ and the overlap region is then where $u = O(1)$ and $t = O\left(\sqrt{\varepsilon}\right)$. Making x_{inner} and x_{outer} agree in the overlap region means keeping u fixed as $\varepsilon \searrow 0$.

The matching condition (3.11) fixes A. We then define an approximate solution $x_a(t)$ by

$$\boxed{x_a(t) = x_{\text{outer}}(t) + x_{\text{inner}}(t) - CL.}$$

The common limit is subtracted because in the intermediate (overlap) region x_{outer} and x_{inner} are equally valid and each is close to the common limit.

Example 3.3 (Continued)

We have

$$x_{\text{outer}}(t) = \exp\left(\frac{1-t}{2}\right) \quad \text{and} \quad x_{\text{inner}}(t) = A\left(1 - \exp\left(-\frac{2t}{\varepsilon}\right)\right),$$

and sonce

$$u = \sqrt{st} = \sqrt{\frac{t^2}{\varepsilon}} = \frac{t}{\sqrt{\varepsilon}}$$

is the intermediate variable we have

$$x_{\text{outer}}\left(\sqrt{\varepsilon}u\right) = e^{\frac{1-\sqrt{\varepsilon}u}{2}} \quad \text{and} \quad x_{\text{inner}}\left(\sqrt{\varepsilon}u\right) = A\left(1 - e^{-\frac{2u}{\sqrt{\varepsilon}}}\right).$$

For matching, we require

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} e^{\frac{1-\sqrt{\varepsilon}u}{2}} = \lim_{\varepsilon \searrow 0} A\left(1 - e^{-\frac{2u}{\sqrt{\varepsilon}}}\right) = CL \\
&\Rightarrow e^{1/2} = A = CL.
\end{aligned}$$

So the composite approximation is

$$\begin{aligned}
x_a(t) &= e^{(1-t)/2} + e^{1/2}\left(1 - e^{-2t/\varepsilon}\right) - e^{1/2} \\
&= e^{1/2}\left(e^{-t/2} - e^{-2t/\varepsilon}\right).
\end{aligned}$$

Note that in the outer region $e^{-2t/\varepsilon}$ is small and so in this region

$$x_a(t) \sim e^{1/2-t/2} \;=\; \text{the outer solution.}$$

In the inner region, $t \ll 1 \Rightarrow \; e^{-t/2} \sim 1$ and so

$$x_a(t) \sim e^{1/2}\left(1 - e^{-2t/\varepsilon}\right) \;=\; \text{the inner solution.}$$

Remark 3.2.

1) *The region where the inner solution is valid contributes to the boundary layer (a term taken from fluid mechanics). We say that the boundary layer has width ε. It is convenient to have this occur at $t = 0$, which can always be achieved by a shift of the origin of the original independent variable.*

2) *The main concern in Singular Perturbation theory is to match an inner solution with an outer solution to obtain an approximation valid over the whole interval. In our examples we do not go beyond zeroth order approximations for either solutions. It is possible to define techniques that include higher order corrections.*

3) The method does not always work, but will for a certain class of problems involving 2^{nd} order linear homogeneous ODEs. See the following theorem.

Theorem 3.1. ***(A Theorem on the Method of Singular Perturbations)*** *Given a linear boundary-value problem of the form*

$$\begin{aligned} &\varepsilon\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0, \quad t \in (0,1), \quad 0 < \varepsilon \ll 1, \\ &x(0) = a, \quad x(1) = b, \end{aligned} \tag{3.12}$$

where p, q are continuous on $[0,1]$ with $p(t) > 0$ for $t \in [0,1]$, there exists a boundary layer at $t = 0$ with outer and inner approximations:

$$x_{outer}(t) = b\exp\left(\int_t^1 \frac{q(\tau)}{p(\tau)}d\tau\right), \quad x_{inner}(t) = A + (a - A)\exp\left(-\frac{p(0)t}{\varepsilon}\right),$$

where

$$A = b\exp\left(\int_0^1 \frac{q(\tau)}{p(\tau)}d\tau\right).$$

Proof. The direct method of perturbation leads to

$$p(t)\dot{x}_0(t) + q(t)x_0(t) = 0$$

for the zeroth-order approximation. Solve this by separating the variables:

$$\frac{\dot{x}_0(t)}{x_0(t)} = -\frac{q(t)}{p(t)},$$

and then integrating from t to 1:

$$[\ln x_0(t)]_t^1 = -\int_t^1 \frac{q(\tau)}{p(\tau)}d\tau, \;\Rightarrow\; \ln\frac{x_0(1)}{x_0(t)} = -\int_t^1 \frac{q(\tau)}{p(\tau)}d\tau,$$

which gives

$$x_0(t) = x_0(1)\exp\left(\int_t^1 \frac{q(\tau)}{p(\tau)}d\tau\right).$$

For the outer solution, the approximate boundary condition is $x_0(1) = b$, so we get

$$x_{\text{outer}}(t) = b\exp\left(\int_t^1 \frac{q(\tau)}{p(\tau)}d\tau\right).$$

For the inner solution, rescale using $s = t/\varepsilon$. The equation becomes (on multiplying through by ε^2)

$$x'' + p(\varepsilon s)x' + \varepsilon q(\varepsilon s)x = 0,$$

where the prime denotes differentiation with respect s.
The zeroth-order approximation (got by setting $\varepsilon = 0$) is

$$x_0'' + p(0)x_0' = 0,$$

with general solution $x_0(s) = A + Be^{-p(0)s}$.
For the inner solution, the appropriate boundary condition is $x_0(0) = a$, which gives $a = A + B$. So $B = a - A$ and the inner solution is

$$x_{\text{inner}}(t) = A + (a - A)e^{-p(0)t/\varepsilon}.$$

All that remains to be done is to show that the constant A is as claimed in the theorem. We do this by using the intermediate variable $u = t/\sqrt{\varepsilon}$ and applying the matching condition:

$$\lim_{\varepsilon\searrow 0}\left\{b\exp\left(\int_{\varepsilon\sqrt{u}}^1 \frac{q(\tau)}{p(\tau)}d\tau\right)\right\} = \lim_{\varepsilon\searrow 0}\left\{A + (a - A)\exp\left(-\frac{p(0)u}{\sqrt{\varepsilon}}\right)\right\}, \tag{3.13}$$

which gives

$$A = b\exp\left(\int_0^1 \frac{q(\tau)}{p(\tau)}d\tau\right),$$

as claimed. □

Remark 3.3.

1) The approximation on $[0,1]$ from considering the inner and outer solutions is then

$$x_a(t) = x_{inner}(t) + x_{outer}(t) - A,$$

where

$$A = b \exp\left(\int_0^1 \frac{q(\tau)}{p(\tau)} d\tau\right).$$

2) Matching is only possible because $p(0) > 0$. However if $p(t) < 0,\ t \in [0,1]$ then there exists a boundary layer at $t = 1$ and again we can match (see Exercise sheet 4, question 3).

We now show two different examples.

Example 3.4. *Use singular perturbation theory and matching to find an approximate solution to the BVP*

$$\left.\begin{array}{l} \varepsilon\ddot{x} + \dot{x} = 2t, \quad t \in (0,1),\ 0 < \varepsilon \ll 1, \\ x(0) = 1,\ x(1) = 1. \end{array}\right\} \tag{3.14}$$

Solution:
Direct method leads to zeroth order approximation:

$$\dot{x}_0 = 2t \quad \Rightarrow \quad x_0(t) = t^2 + A.$$

Impose $x_0(1) = 1 \ \Rightarrow\ A = 0$ and hence $x_0(t) = t^2$.
Assume that there exists a boundary layer at $t = 0$ and rescale near $t = 0$.
Using $s = t/\delta$ to get

$$\frac{\varepsilon}{\delta^2}x'' + \frac{1}{\delta}x' = 2\delta s, \tag{3.15}$$

where dashes denote differentiating with respect to s.
Balancing two of the terms in the equation:

a)
$$\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \ \Rightarrow\ \delta \sim \varepsilon \quad \text{(possible)}.$$

b)
$$\frac{1}{\delta} \sim 2\delta \ \Rightarrow\ \delta \sim 1 \quad \text{(no change and hence reject)}.$$

c)
$$\frac{\varepsilon}{\delta^2} \sim 2\delta \ \Rightarrow\ \delta \sim \varepsilon^{1/3} \quad \text{(possible)}.$$

First we try c). Setting $\delta = \varepsilon^{1/3}$ in (3.15) gives

$$\varepsilon^{2/3}x'' + x' = 2\varepsilon^{2/3}s.$$

Setting $\varepsilon = 0$ gives the zeroth order approximation:

$$x_0' = 0$$

since this is a first order equation and the original ODE is second order we reject this choice of δ.
Now we try a). Taking $\delta = \varepsilon$ in (3.15) gives

$$x'' + x' = 2\varepsilon^2 s.$$

Setting $\varepsilon = 0$ gives the zeroth order approximation

$$x_0'' + x_0' = 0 \quad \Rightarrow \quad x_0(s) = A + Be^{-s}.$$

Imposing the boundary condition $x_0(0) = 1 \;\Rightarrow\; A = 1 - B$,

$$\Rightarrow\; x_0(s) = 1 - B(1 - e^{-s})$$

i.e., we have

$$x_{\text{inner}}(t) = x_0(s) = x_0(t/\varepsilon) = 1 - B(1 - e^{-t/\varepsilon}).$$

We recall that the outer solution is

$$x_{\text{outer}}(t) = t^2.$$

For matching we use an intermediate scale $u = \sqrt{ts} = t/\sqrt{\varepsilon}$.
The overlap region is $t = O(\sqrt{\varepsilon}u)$ where $u = O(1)$ and we want the outer and inner solutions to 'match' in this overlap region, so we require

$$\lim_{\varepsilon\searrow 0} x_{\text{outer}}(\sqrt{\varepsilon}u) = \lim_{\varepsilon\searrow 0} x_{\text{inner}}(\sqrt{\varepsilon}u) := CL$$

$$\Rightarrow\; \lim_{\varepsilon\searrow 0}(\sqrt{\varepsilon}u)^2 = \lim_{\varepsilon\searrow 0}\left(1 - B\left(1 - e^{-u/\sqrt{\varepsilon}}\right)\right) = CL$$

$$\Rightarrow\; 0 = 1 - B = CL$$

i.e., $B = 1$ is the matching condition and $CL = 0$.
Hence our zeroth order approximation is

$$\begin{aligned} x_a(t) &= x_{\text{outer}}(t) + x_{\text{inner}}(t) - CL \\ &= t^2 + (1 - (1 - e^{-t/\varepsilon})) - 0 \\ &= t^2 + e^{-t/\varepsilon}. \end{aligned}$$

Example 3.5. *Use singular perturbation theory and matching to find an approximate solution to the BVP*

$$\left.\begin{array}{l} \varepsilon\ddot{x} + t\dot{x} - tx = 0,\; 0 < \varepsilon \ll 1, \quad t \in (0,1), \\ x(0) = 0,\; x(1) = e. \end{array}\right\} \tag{3.16}$$

Solution:
Direct method $\Rightarrow$ zeroth order equation is

$$t\dot{x}_0 - tx_0 = 0 \quad\Rightarrow\; \dot{x}_0 - x_0 = 0$$
$$\Rightarrow x_0(t) = Ae^t.$$

Applying the right hand (outer) boundary condition $x_0(1) = e$ gives $A = 1$ and hence we have an outer solution

$$x_{\text{outer}}(t) = e^t.$$

To find an inner solution we rescale with

$$s = \frac{t}{\delta}$$

and (3.16) becomes

$$\frac{\varepsilon}{\delta^2}x'' + \frac{\delta s}{\delta}x' - s\delta x = 0. \tag{3.17}$$

Possible balances are :-

a)

$$\frac{\varepsilon}{\delta^2} \sim 1 \;\Rightarrow\; \delta = \sqrt{\varepsilon}.$$

Taking $\delta = \sqrt{\varepsilon}$ in (3.17) gives $x'' + sx' - s\sqrt{\varepsilon}x = 0$, which looks OK since it is second order.

b)

$$\frac{\varepsilon}{\delta^2} \sim \delta \;\Rightarrow\; \delta \sim \varepsilon^{1/3}.$$

Taking $\delta = \varepsilon^{1/3}$ in (3.17) gives $\varepsilon^{1/3}x'' + sx' - s\varepsilon^{1/3}x = 0$.
Setting $\varepsilon = 0$ gives a first order equation, so we reject this choice of δ.

c) $\delta \sim 1, \Rightarrow \delta = 1$ which gives no change and so we reject this choice of δ.

Hence we use a).
Setting $\varepsilon = 0$ gives the zeroth order approximation

$$x_0'' + sx_0' = 0.$$

Setting $u = x_0'$ gives

$$u' + su = 0 \quad \Rightarrow \quad \int \frac{du}{u} = -\int s ds$$
$$\Rightarrow \ln u = -\frac{s^2}{2} + C$$
$$\Rightarrow u = x_0' = Ae^{-s^2/2}$$
$$\Rightarrow x_0(s) = x_0(0) + \int_0^s Ae^{-\tau^2/2} d\tau.$$

Applying the inner boundary condition $x_0(0) = 0$ gives

$$x_{\text{inner}}(t) = A \int_0^{t/\sqrt{\varepsilon}} e^{-\tau^2/2} d\tau.$$

To match with the outer solution we require 'matching' in the overlap region determined by the intermediate scale

$$u = \sqrt{ts} = \frac{t}{\varepsilon^{1/4}}.$$

Hence the overlap region is $t = O(\varepsilon^{1/4}u)$, $u = O(1)$. The matching condition is then

$$\lim_{\varepsilon \searrow 0} x_{\text{inner}}(\varepsilon^{1/4}u) = \lim_{\varepsilon \searrow 0} x_{\text{outer}}(\varepsilon^{1/4}u) := CL$$

i.e., the matching condition is

$$\lim_{\varepsilon \searrow 0} A \int_0^{\frac{u}{\varepsilon^{1/4}}} e^{-\tau^2/2} d\tau = \lim_{\varepsilon \searrow 0} e^{\varepsilon^{1/4}u} = CL$$
$$\Rightarrow A \int_0^{\infty} e^{-\tau^2} 2d\tau = 1 = CL$$
$$\Rightarrow A\sqrt{\frac{\pi}{2}} = 1 = CL$$

and hence

$$A = \sqrt{\frac{2}{\pi}} \quad \text{and} \quad CL = 1$$
$$\Rightarrow x_a(t) = \sqrt{\frac{2}{\pi}} \int_0^{t/\sqrt{\varepsilon}} e^{-\tau^2/2} d\tau + e^t - 1.$$

(The point of this example is that the definition of the overlap region is different to previous example.)