Methods of applied mathematics

Lecture notes

Autumn Term 2008

Contents

1 Dimensional analysis and scaling

1.1 Physical quantities and their measurement

By a *physical quantity* we mean anything that can be measured by some strictly defined measuring process.

Example 1.1.

- *Temperature by means of a thermometer.*
- *Time by means of a clock.*

A physical quantity may be variable or constant. For example a function of time such as the temperature of a cooling cup of hot tea or a universal constant such as the speed of light. To measure a physical quantity, we need a *unit of measurement*. This allows us to attach a real number to the quantity and results in a *magnitude* relative to the unit employed.

Example 1.2.

- *13 feet (length of rod).*
- *14.5 degrees Centigrade (air temperature).*
- *200.37 Km/hr (speed of a car).*
- $33\frac{1}{3}$ *revolutions per minute (angular velocity of a turntable).*

If the system of units is changed then magnitudes also change. If
new unit $= \frac{1}{\lambda} \times$ old unit ($\lambda > 0$), then, new magnitude $= \lambda \times$ old magnitude.

Example 1.3. *If the old unit is the foot and the new unit is the centimetre then since 1 centimetre is approximately* $\frac{1}{30}$ *feet then* $\lambda = 30$.

Note that we are assuming that 0 in one unit is 0 in another, unlike when one changes from degrees centigrade to degrees Fahrenheit.

1.1.1 Dimensions and units

Roughly speaking the *dimensions* of a physical quantity correspond to how it is defined or how it is measured. Any physical theory regards certain dimensions as *fundamental* and others as *derived*.

Example 1.4. *Speed has dimensions length/time. Here length and time are regarded as fundamental, whereas the dimension of speed is derived.*

Notation

- We use uppercase letters to denote fundamental dimensions.
  e.g. M for mass, L for length, T for time, Θ for temperature.
- To indicate the dimensions of a physical quantity we enclose it in square brackets.
  e.g. [speed], [g], $[\frac{1}{2}mv^2]$.
- Dimensional equations are used to relate and manipulate the dimensions of physical quantities.
  Some examples of useful physical physical quantities:
  - [velocity]=[length/time]=LT^{-1}
  - [acceleration]=[velocity/time]=LT^{-2}
  - [angular velocity]=[angle/time]=T^{-1}
  - [frequency]=T^{-1}
  - [momentum]=[mass $\times$ velocity]=MLT^{-1}

- [force]=[momentum/time]=MLT^{-2}
- [density]=[mass/volume]=ML^{-3}
- [energy]=[force× distance]=ML^2T^{-3}
- [power]=[energy/time]=ML^2T^{-3}
- [pressure]=[force/area]=$ML^{-1}T^{-2}$
- [heat]=[energy]=ML^2T^{-2}
- [heat capacity]=[heat/degree]=$ML^2T^{-2}\Theta^{-1}$
- [specific heat]=[heat/(mass× degree)]=$L^2T^{-2}\Theta^{-1}$.

- We use 1 to denote the dimensions of a dimensionless quantity.
  e.g. [angle] = [arc-length/radius] = L/L =1.

If a theory is extended then more (or possibly fewer) fundamental dimensions may be needed.

- For geometry, L = length is sufficient.
- For kinematics we need L = length and T = time.
- For simple mechanics, we need M = mass, L = length and T = time.
- In order to extend mechanics to include thermal properties of bodies we add Θ for absolute temperature to the fundamental dimensions M, L and T.
- In order to extend the theory to include electrical properties we add Q = electric charge.

1.2 Change of Units

Suppose that in some theory the fundamental dimensions are

$$L_1, L_2, \cdots, L_n$$

and that each has a corresponding unit for measuring it. Let

$$l_1, l_2, \cdots, l_n$$

be the magnitudes of quantities with these fundamental dimensions relative to the corresponding units. If the units are changed then we obtain a new set of magnitudes

$$\bar{l}_k = \lambda_k l_k,\ k = 1, 2, \cdots, n \quad (\lambda_k > 0) \tag{1.1}$$

where λ_k are scaling factors associated with the change of units.

Now consider a physical quantity q whose dimensions are derived. We have that the dimensions of q can be written in terms of the fundamental dimensions in the form

$$[q] = L_1^{b_1} L_2^{b_2} \cdots L_n^{b_n}.$$

Thus its magnitude will change according to the formula

$$\bar{q} = \lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_n^{b_n} q. \tag{1.2}$$

Example 1.5. *Let q be a speed in miles per hour and $\bar{q}$ be the same speed in feet per second then*

$$\bar{q} = 5280 \ \times \ 3600^{-1} q$$

since 1 mile is 5280 feet and 1 hour is 3600 seconds and $[q] = LT^{-1}$.

1.3 Physical Laws

An equation that involves the magnitudes $q_1, q_2, \cdots, q_m$ of m physical quantities will be referred to as a *physical law.*

Example 1.6. *Let x be the distance in feet of an object fallen from rest under gravity and let t be the time in seconds that have elapsed during the fall, then the physical law is approximately*

$$x = 16t^2 \tag{1.3}$$

Note that the law (1.3) is not (in general) correct if the units are changed. If cm are used instead of feet, the law is $\bar{x} = 490\bar{t}^2$ approximately and hence the law depends on the units used.

Definition 1.1 (Unit free physical laws)**.** *Let*

$$f(q_1, \cdots, q_m) = 0 \tag{1.4}$$

be a physical law involving the magnitudes q_k of m physical quantities. Under a change of units characterized by equation (1.1) *each magnitude q_k will transform according to an equation of the form* (1.2) *(where the exponents $b_1, \cdots, b_n$ are determined by the dimensions of q_k). The physical law* (1.4) *is* **unit free***, if for all scaling factors $\lambda_1, \cdots, \lambda_n$ $(\lambda_k > 0)$ we have*

$$f(\bar{q}_1, \cdots, \bar{q}_m) = 0 \Leftrightarrow f(q_1, \cdots, q_m) = 0.$$

Example 1.7. *We remarked that $x = 16t^2$ is not unit free. However suppose we introduce gravity g (measured in feet/sec^2) into the law and write it as*

$$x = \frac{1}{2}gt^2, \tag{1.5}$$

or

$$f(x, t, g) = 0,$$

where

$$f(x, t, g) \equiv x - \frac{1}{2}gt^2.$$

Is this law unit free?

Solution: Change units so that $\bar{x} = \lambda_1 x$ and $\bar{t} = \lambda_2 t$ $(\lambda_i > 0,\ i = 1, 2)$. Then because $[g] = LT^{-2}$, we have $\bar{g} = \lambda_1\lambda_2^{-2}g$ and

$$\begin{aligned} f(\bar{x}, \bar{t}, \bar{g}) &= \bar{x} - \frac{1}{2}\bar{g}\bar{t}^2 \\ &= \lambda_1 x - \frac{1}{2}(\lambda_1\lambda_2^{-2}g)(\lambda_2 t)^2 \\ &= \lambda_1(x - \frac{1}{2}gt^2) \\ &= \lambda_1 f(x, t, g). \end{aligned}$$

So

$$f(\bar{x}, \bar{t}, \bar{g}) = 0 \Leftrightarrow f(x, t, g) = 0$$

and hence the law (1.5) is unit free.

Remark 1.1.

1) *In a unit free law any numbers which appear are dimensionless. (For example, the "1/2" in $x = (1/2)gt^2$.)*

2) *If a physical law is unit free then the formula is valid whatever units are used for the fundamental dimensions. Thus it makes sense to regard the law as a law connecting the physical quantities themselves rather than their magnitudes relative to some system of units.*

1.4 Buckingham Pi Theorem

From x, t and g in Example 1.7 we can form the dimensionless quantity

$$\pi_1 = \frac{x}{gt^2},$$

since

$$[\pi_1] = \frac{L}{LT^{-2}} \frac{1}{T^2} = 1.$$

Furthermore the unit free law (1.5) can be written as

$$\pi_1 = \frac{1}{2}.$$

However from x and t alone we cannot form a dimensionless quantity and there is no way that we can express the foot/second version (1.3) (or cm/sec version, etc) in terms of dimensionless quantities.

The ability to express unit free laws in terms of dimensionless quantities is the essential content of the Buckingham Pi Theorem.
Before we state the Buckingham Pi Theorem we will recall the definition of the **rank** of a matrix.

- The row (column) **rank** of a matrix is the number of rows (columns) that are **linearly independent**.
- The rows (columns) in a matrix are **linearly independent** if they can not be written as a linear combination of the other rows (columns).
- The row rank is equal to the column rank and they are simply called the rank of a matrix.
- The easiest way to compute the rank is by Gaussian elimination. The row-echelon form of a matrix A produced by the Gauss algorithm has the same rank as A, and its rank can be read off as the number of non-zero rows.

Theorem 1.1 (Buckingham Pi Theorem). *Let*

$$f(q_1, q_2, \cdots, q_m) = 0 \tag{1.6}$$

be a unit free physical law that relates the positive magnitudes $q_1, q_2, \cdots, q_m$ of m physical quantities. Let the dimensions of these be given by

$$[q_k] = L_1^{a_{1k}} L_2^{a_{2k}} \cdots L_n^{a_{nk}} \tag{1.7}$$

relative to a set of fundamental dimensions $L_1, L_2, \cdots, L_n$.
Let A be the $n \times m$ matrix with coefficients $\{a_{jk}\}$. We say that A is the dimension matrix associated with the law whose elements are the exponents occurring in (1.7).
If r =rank of A then there exists $(m - r)$ independent dimensionless quantities $\pi_1, \pi_2, \cdots, \pi_{m-r}$ that can be formed from combinations of $q_1, \cdots, q_m$ and the physical law is equivalent to one of the form

$$F(\pi_1, \pi_2, \cdots, \pi_{m-r}) = 0 \tag{1.8}$$

which involves only the dimensionless quantities.

Proof. We try to find a dimensionless quantity of the form

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m} \qquad \text{with } [\pi] = 1. \tag{1.9}$$

Since

$$[\pi] = [q_1]^{\alpha_1} \cdots [q_k]^{\alpha_k} \cdots [q_m]^{\alpha_m} \qquad \text{with } \ [q_k]^{\alpha_k} = \Pi_{j=1}^n L_j^{a_{jk}\alpha_k}$$

we have

$$[\pi] = \prod_{j=1}^n L_j^{\sum_{k=1}^m a_{jk}\alpha_k} = \prod_{j=1}^n L_j^0 = 1.$$

Hence for each $j = 1, 2, \cdots, n$ we have

$$\sum_{k=1}^{m} a_{jk}\alpha_k = 0 \;\Leftrightarrow\; A\underline{\alpha} = 0 \quad \text{where} \quad \underline{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

Thus the condition that π be dimensionless then leads to

$$A\underline{\alpha} = \underline{0}, \tag{1.10}$$

where $A = \{a_{jk}\}$ is the dimension matrix and $\underline{\alpha} = [\alpha_1 \; \alpha_2 \; \cdots \; \alpha_m]^T$ is the column vector of exponents occurring in (1.9). We then use a theorem from linear algebra to justify the claim of the Buckingham Pi Theorem that there are $(m-r)$ independent dimensionless quantities $\pi_1, \pi_2, \cdots, \pi_{m-r}$ that can be formed from $q_1, q_2, \cdots, q_m$. These quantities are obtained by taking $(m-r)$ independent solutions of (1.10), which exist because the solution space of (1.10) has dimension $(m-r)$.
It remains to show that the physical law (1.6) is equivalent to one of the form

$$F(\pi_1, \pi_2, \cdots, \pi_{m-r}) = 0,$$

involving only the dimensionless quantities. We do this by using the fact that the given law (1.6) is unit-free and by drawing on the method of solving (1.10) by Gaussian elimination.
Gaussian elimination leads to a row-reduced echelon matrix E that is row-equivalent to A. The matrix E has r non-zero rows (as rank $A = r$), each with one as its leading non-zero term, with zeros above and below this one (as E is row-reduced). If $r < n$, then the $(n-r)$ rows of zeros can be discarded, as they contain no information, and the resulting matrix (which we still denote E) is then $r \times m$. Without loss of generality, we can then reorder the magnitudes $q_1, \cdots, q_m$ so that the columns containing a single one with zeros above and below (as generated by the row-reduction) occupy the first r positions. Then E has the form

$$E = [I_r|(-B)], \tag{1.11}$$

where I_r is an $r \times r$ unit matrix and B is an $r \times (m-r)$ matrix. (The minus sign is for later convenience.) The general solution to (1.11) is then given by regarding the $(m-r)$ unknowns $\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_m$ as arbitrary parameters, with the r unknowns $\alpha_1, \alpha_2, \cdots, \alpha_r$ determined in terms of them by (1.11):

$$\alpha_j = \sum_{k=1}^{m-r} b_{jk}\alpha_{r+k} \quad (j = 1, \cdots, r),$$

where $\{b_{jk}\} = B$. Taking $\alpha_{r+1} = 1$ and the other $\alpha_{r+k} = 0$, gives the solution

$$\alpha_1 = b_{11},\; \alpha_2 = b_{21},\; \cdots,\; \alpha_r = b_{r1},$$
$$\alpha_{r+1} = 1,\; \alpha_{r+2} = 0,\; \alpha_m = 0,$$

which yields the dimensionless quantity

$$\pi_1 = (q_1^{b_{11}} q_2^{b_{21}} \cdots q_r^{b_{r1}})q_{r+1}.$$

Similarly, taking $\alpha_{r+2} = 1$ and the other $\alpha_{r+k} = 0$, we get

$$\pi_2 = (q_1^{b_{12}} q_2^{b_{22}} \cdots q_r^{b_{r2}})q_{r+2},$$

and we can proceed in this way to get $(m-r)$ dimensionless quantities, the general one of which is

$$\pi_j = (q_1^{b_{1j}} q_2^{b_{2j}} \cdots q_r^{b_{rj}})q_{r+j}. \tag{1.12}$$

So, for $j = 1, \cdots, m-r$, we can express q_{r+j} in terms of π_j and $q_1, \cdots, q_r$:

$$q_{r+j} = \pi_j(q_1^{b_{1j}} q_2^{b_{2j}} \cdots q_r^{b_{rj}})^{-1}. \tag{1.13}$$

Hence, we can write the given law (1.6) as

$$f\left(q_1, \cdots, q_r, \pi_1(q_1^{b_{11}} q_2^{b_{21}} \cdots q_r^{b_{r1}})^{-1}, \cdots, \pi_{m-r}(q_1^{b_{1,m-r}} q_2^{b_{2,m-r}} \cdots q_r^{b_{r,m-r}})^{-1}\right) = 0. \qquad (1.14)$$

The equivalent law $F(\pi_1, \cdots, \pi_{m-r}) = 0$ is then obtained by what amounts to setting $q_1 = q_2 = \cdots = q_r = 1$ in (1.14). The argument depends on the fact that the given law is unit-free, and is as follows. Because the law is unit-free, we have

$$f(q_1, \cdots, q_m) = 0 \Leftrightarrow f(\bar{q}_1, \cdots, \bar{q}_m) = 0,$$

where $\bar{q}_1, \cdots, \bar{q}_m$ are new magnitudes given by equations of the form (1.2). These new magnitudes arise when magnitudes having fundamental dimensions change according to (1.1). We shall show that for any set of magnitudes $q_1, \cdots, q_m$ with each $q_j > 0$, there exist conversion factors $\lambda_1, \cdots, \lambda_n$ that result in

$$\bar{q}_1 = \bar{q}_2 = \cdots = \bar{q}_r = 1. \qquad (1.15)$$

We shall then have

$$f(q_1, \cdots, q_m) = 0 \Leftrightarrow f(\bar{q}_1, \cdots, \bar{q}_m) = 0 \Leftrightarrow f(1, \cdots, 1, \pi_1, \cdots, \pi_{m-r}) = 0,$$

and defining F by

$$F(\pi_1, \cdots, \pi_{m-r}) = f(1, \cdots, 1, \pi_1, \cdots, \pi_{m-r})$$

will then complete the proof.
All that remains to be done is to show that we can indeed find conversion factors λ_j such that (1.15) holds.
Since our magnitudes have dimensions given by (1.7) and corresponding to (1.2) we have

$$\bar{q}_j = \lambda_1^{a_{1j}} \lambda_2^{a_{2j}} \cdots \lambda_n^{a_{nj}} q_j \quad (j = 1, \cdots, m).$$

For any given $q_1, \cdots, q_m$, we need $\lambda_1, \cdots, \lambda_n$ such that (1.15) holds. So, we need $\lambda_1, \cdots, \lambda_n$ such that

$$1 = \lambda_1^{a_{1j}} \lambda_2^{a_{2j}} \cdots \lambda_n^{a_{nj}} q_j \quad (j = 1, \cdots, r).$$

Taking logarithms, we see that we need values of $\lambda_1, \cdots, \lambda_n$ satisfying

$$a_{1j} \ln \lambda_1 + a_{2j} \ln \lambda_2 + \cdots + a_{nj} \ln \lambda_n = -\ln q_j \quad (j = 1, \cdots, r).$$

(Note that it is here that we use the restriction that each q_k is positive.) That is, we need to solve the inhomogeneous system

$$CX = D, \qquad (1.16)$$

where

$$X = [\ln \lambda_1, \ \ln \lambda_2, \ \cdots, \ \ln \lambda_n]^T, \ D = [-\ln q_1, \ -\ln q_2, \ \cdots \ -\ln q_r]^T$$

and $C = A_r^T$. Here A_r denotes the $n \times r$ matrix comprising of the first r columns of the dimension matrix A.
Now, we ordered the magnitudes q_k so that by means of row operators we obtained

$$A_r \longrightarrow \left[\frac{I_r}{O}\right],$$

where I_r is an $r \times r$ unit matrix and O is a matrix of zeros. So

$$\operatorname{rank} C = \operatorname{rank} A_r^T = \operatorname{rank} A_r = \operatorname{rank} I_r = r,$$

showing that C has full rank. This implies that (1.16) can always be solved to obtain conversion factors $\lambda_1, \cdots, \lambda_n$ that, for any $q_1, \cdots, q_m$ (with each $q_k > 0$), produce the result (1.15), as required. □

Remark 1.2. *The exponents α_i for $i = 1, 2, \cdots, m$ of each of the $(m - r)$ dimensionless quantities*

$$\pi_j = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m} \quad j = 1, 2, \cdots, m - r$$

can be obtained by finding $m - r$ linearly independent solutions of $A\underline{\alpha} = \underline{0}$.

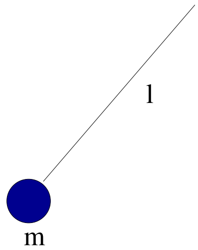

Figure 1: A simple pendulum.

Example 1.8. *Is it true to assume that the period τ of a simple pendulum depends on its length l, the mass m of its 'bob' and the acceleration due to gravity g? (See Figure 1).*

Solution:
Let us assume a unit free law of the form

$$f(\tau, l, m, g) = 0. \tag{1.17}$$

Since the dimensions are

$$[\tau] = T, [l] = L, [m] = M, [g] = LT^{-2}$$

we have the following dimension matrix

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccc} m & l & \tau & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \end{array} = A$$

with rank A=3.
From the Buckingham Pi theorem we have that the unit-free law (1.17) is equivalent to one involving $m - r$ dimensionless quantities. Since $m = 4$ and $r = 3$ there exists 1 dimensionless quantity

$$\pi = m^{\alpha_1} l^{\alpha_2} \tau^{\alpha_3} g^{\alpha_4}$$

such that the unit free law (1.17) is equivalent to a unit free law of the form

$$F(\pi) = 0$$

involving one dimensionless quantity $\pi = m^{\alpha_1} l^{\alpha_2} \tau^{\alpha_3} g^{\alpha_4}$.
To find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ we find a basis for the solution space of the homogeneous system $A\underline{\alpha} = 0$

$$\begin{array}{llllr} \alpha_1 & & & & = 0, \\ & \alpha_2 & & +\alpha_4 & = 0, \\ & & \alpha_3 & -2\alpha_4 & = 0. \end{array}$$

If we choose $\alpha_2 = -1$ it follows that $\alpha_1 = 0$, $\alpha_3 = 2$ and $\alpha_4 = 1$ and hence

$$\underline{\alpha} = [0, -1, 2, 1] \tag{1.18}$$

giving

$$\begin{aligned} \pi &= m^{\alpha_1} l^{\alpha_2} \tau^{\alpha_3} g^{\alpha_4} \\ &= m^0 l^{-1} \tau^2 g \\ &= \frac{\tau^2 g}{l}, \end{aligned}$$

as a *dimensionless quantity.*

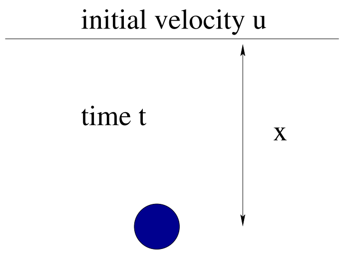

Figure 2: Descent under gravity

Thus the law for period is equivalent to one of the form

$$F(\tau^2 g/l) = 0,$$

where $F(\cdot)$ is an unknown function, so the law does not depend on m as we had assumed. From this we can infer that

$$\frac{\tau^2 g}{l} = \text{constant} = c, \text{ say (a zero of } F),$$
$$\text{i.e. } \tau = c\sqrt{1/g}$$

(Actually the simplest theory involving Newton's law of motion $\Rightarrow \ \tau = 2\pi\sqrt{l/g}$.)

Example 1.9. *Show that the law for descent under gravity for a body which has an initial downward velocity u (see Figure 2) can be written in terms of two dimensionless quantities.*

Solution: We assume, a unit free law of the form

$$f(x, t, g, u) = 0. \tag{1.19}$$

Then the dimension matrix is

$$\begin{matrix} & x & t & g & \text{u} \\ L & 1 & 0 & 1 & 1 \\ T & 0 & 1 & -2 & -1 \end{matrix} = A$$

with rank A=2. Since the law is unit free we can apply the Buckingham Pi theorem which says that there are $m - r = 4 - 2 = 2$ dimensionless quantities π_1 and π_2, such that we have an equivalent unit free law to (1.19) of the form

$$F(\pi_1, \pi_2) = 0.$$

Furthermore π_1 and π_2 are chosen by taking two independent solutions of $A\underline{\alpha} = 0$

$$\Rightarrow \quad \begin{array}{rrrrl} \alpha_1 & & +\alpha_3 & +\alpha_4 & = 0, \\ & \alpha_2 & -2\alpha_3 & -\alpha_4 & = 0. \end{array} \tag{1.20}$$

If we choose $\alpha_3 = 1$ and $\alpha_4 = 0$ it follows that $\alpha_1 = -1$ and $\alpha_2 = 2$. Whereas choosing $\alpha_3 = 0$ and $\alpha_4 = 1$ it follows that $\alpha_1 = -1$ and $\alpha_2 = 1$. and hence two independent solutions are:

$$\underline{\alpha} = [-1\ 2\ 1\ 0]^T = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]^T \quad \text{and} \quad \underline{\alpha} = [-1\ 1\ 0\ 1]^T = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]^T.$$

Since

$$\pi = x^{\alpha_1} t^{\alpha_2} g^{\alpha_3} u^{\alpha_4}$$

we have that

$$\pi_1 = x^{-1} t^2 g^1 u^0 = t^2 g/x \quad \text{and} \quad \pi_2 = x^{-1} t^1 g^0 u^1 = ut/x.$$

Hence the Buckingham Pi theorem says that there is an equivalent law to (1.19) of the form

$$F(\pi_1, \pi_2) \equiv F\left(\frac{gt^2}{x}, \frac{ut}{x}\right) = 0. \tag{1.21}$$

The actual law is given by solving

$$\ddot{x} = g, \qquad x(0) = 0,\ \dot{x}(0) = u.$$

Integrating $\ddot{x} = g$ once with respect to t gives

$$\dot{x} = gt + c$$

and since $\dot{x}(0) = u$ we have $\dot{x} = gt + u$. Integrating again and using $x(0) = 0$ then gives

$$x = ut + \frac{1}{2}gt^2 \quad \Rightarrow \quad 1 = \frac{ut}{x} + \frac{1}{2}\frac{gt^2}{x} = \pi_2 + \frac{1}{2}\pi_1$$
$$\Rightarrow \quad \frac{1}{2}\pi_1 + \pi_2 - 1 = 0.$$

Hence the F in (1.21) is given by $F(\pi_1, \pi_2) = \pi_1/2 + \pi_2 - 1$.

Remark 1.3. *In the general case where A is $n \times m$ and rank $A = r$ the solution space has dimension $(m - r)$ and we have $(m - r)$ basis vectors*

$$\underline{\alpha}_1, \cdots, \underline{\alpha}_{m-r} \quad \textit{(say)}$$

each of which holds a dimensionless combination of physical quantities that yield $\pi_1, \cdots, \pi_{m-r}$.
For the equivalent law we can use any set of $(m - r)$ independent solutions in order to yield our dimensionless quantities $\pi_1, \cdots, \pi_{m-r}$ and an equivalent law $F(\pi_1, \pi_2, \cdots, \pi_{m-r}) = 0$.

Example 1.9 (Revisited)
Set $\tilde{\pi}_1 = \pi_1$ and find a dimensionless quantity $\tilde{\pi}_1$ that is different to π_2 such that (1.19) can be written in the form

$$\widetilde{F}(\tilde{\pi}_1, \tilde{\pi}_2) = 0.$$

Solution:
Choosing $\alpha_3 = 1$ and $\alpha_4 = 1$ in (1.20) yields

$$\underline{\alpha} = [-2\ 3\ 1\ 1]^T$$

and hence two independent solutions are

$$\underline{\alpha} = [-1\ 2\ 1\ 0]^T = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]^T \quad \text{and} \quad \underline{\alpha} = [-2\ 3\ 1\ 1]^T = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]^T$$

which yield the dimensionless quantities

$$\tilde{\pi}_1 = x^{-1}t^2g^1u^0 = \frac{t^2g}{x} = \pi_1 \quad \text{and} \quad \tilde{\pi}_2 = x^{-2}t^3g^1u^1 = \frac{ugt^3}{x^2} = \frac{gt^2}{x}\cdot\frac{ut}{x} = \pi_1\pi_2.$$

The Buckingham Pi theorem says that there exists an equivalent law to (1.19) of the form

$$\widetilde{F}(\tilde{\pi}_1, \tilde{\pi}_2) = 0 \quad \Rightarrow \quad 0 = \widetilde{F}(\pi_1, \pi_1\pi_2).$$

By setting $F(z_1, z_2) := \widetilde{F}(z_1, z_1z_2)$ we have consistency between the two laws.

Remark 1.4. *When a physical law is written in the form*

$$f(q_1, q_2, \cdots, q_m) = 0,$$

we can always rewrite it in the form

$$q_m = g(q_1, \cdots, q_{m-1})$$

provided

$$\frac{\partial f}{\partial x_m}(q_1, \cdots, q_m) \neq 0,$$

(Implicit Function Theorem - Function of Several Variables).

Example 1.10. *Suppose one wishes to determine the power P that must be applied to keep a ship of length l moving at a constant speed V. It seems reasonable to assume that P depends on the density of water ρ, the acceleration due to gravity g and the kinematic viscosity of water ν (measured in length squared per unit time), as well as on l and V. Show that any unit-free law connecting these quantities is equivalent to one of the form*

$$\frac{P}{\rho l^2 V^3} = h(Fr, Re), \tag{1.22}$$

where Fr and Re denote the Froude and Reynolds numbers defined by

$$Fr = \frac{V}{\sqrt{lg}}, \qquad Re = \frac{Vl}{\nu}.$$

Solution:
Assume a unit-free law of the form $g(P, l, \rho, g, \nu, V) = 0$. The dimension matrix A is

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccccc} P & l & \rho & g & \nu & V \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & -3 & 1 & 2 & 1 \\ -3 & 0 & 0 & -2 & -1 & -1 \end{pmatrix} \end{array} = A.$$

Subtracting $2R_1$ from R_2 and adding $3R_1$ to R_3 gives

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccccc} P & l & \rho & g & \nu & V \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 1 & 2 & 1 \\ 0 & 0 & 3 & -2 & -1 & -1 \end{pmatrix} \end{array}$$

so rank A =3 and there are $6 - 3 = 3$ independent solutions. A check verifies that the following are solutions: $\underline{\alpha}_1 = [1\ -2\ -1\ 0\ 0\ -3]^T$, $\underline{\alpha}_2 = [0\ -\frac{1}{2}\ 0\ -\frac{1}{2}\ 0\ 1]^T$ and $\underline{\alpha}_3 = [0\ 1\ 0\ 0\ -1\ 1]^T$ (note that these solutions are chosen by looking at the exponents of P, l, ρ, g, ν and V in Fr, Re and $P/(\rho l^2 V^3)$) . They are clearly independent and yield the dimensionless quantities

$$\pi_1 = \frac{P}{\rho l^2 V^3}, \quad \pi_2 = \frac{V}{\sqrt{lg}}, \quad \pi_3 = \frac{Vl}{\nu}$$

and the equivalent law $G(\pi_1, \pi_2, \pi_3) = 0$ which can be written as $\pi_1 = h(\pi_2, \pi_3)$, i.e. as (1.22).

Remark 1.5. *The above solutions $\underline{\alpha}_1 = [1\ -2\ -1\ 0\ 0\ -3]^T$, $\underline{\alpha}_2 = [0\ -\frac{1}{2}\ 0\ -\frac{1}{2}\ 0\ 1]^T$ and $\underline{\alpha}_3 = [0\ 1\ 0\ 0\ -1\ 1]^T$ are independent since the (6×3) matrix*

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1/2 & 1 \\ -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1 \\ -3 & 1 & 1 \end{pmatrix}$$

can be transformed using row operations into one which contains the (3×3) identity matrix.

1.5 Scaling

An *initial value problem* (IVP) or a *boundary value problem* (BVP) comprises of a differential equation (or a system of equations) together with sufficient initial or boundary conditions to ensure that a unique solution exists. (Equations could be ODEs or PDEs.)
We are interested in problems where the variables appearing in the equation together with the parameters or coefficients are physical quantities with dimensions.
The basic idea of scaling is to take each variable q and find a *characteristic value* q_c with which it can be compared and then to replace q by the dimensionless variable $\bar{q} := q/q_c$.
The characteristic values q_c act as "natural" units and their use should result in scaled magnitudes $\bar{q}$ which are of moderate size.

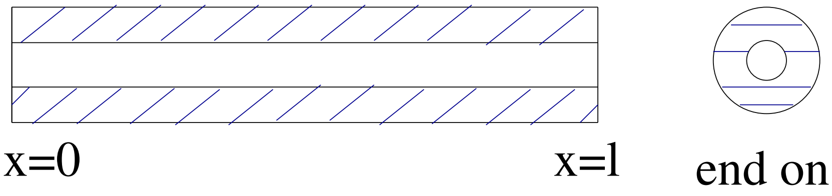

Figure 3:

Example 1.11. ***(The derivation of the one dimensional heat equation)***
Derive a model for the temperature distribution in a rod made up of some homogeneous material (see Figure 3) where except for at the end-points, the rod is insulated. In the model assume that heat flows in $x-$direction only and that the temperature is constant in any cross section (which has area A).

Solution:
Let u denote the temperature and by the assumption that heat flows in $x-$direction only we have $u = u(x,t)$, where t is time.
Since the rod is made up of some homogeneous material we have that

C_v = Specific heat at constant volume ($\equiv$ amount of heat energy required to raise one unit mass one degree of temperature (e.g. calories/grams$\times$°C)) and ρ = density of material (=mass/unit volume). Here C_v and ρ are constants.

If we consider a thin slice of the rod with width δx then the amount of heat energy in the slice is approximately $C_v(\rho A \delta x)u(x,t)$. Hence for any region $x_1 < x < x_2$ of rod (with x_1 and x_2 in $(0,l)$) the total amount of heat energy in this region is $\int_{x_1}^{x_2} C_v \rho A u(x,t)dx$.

On the other hand there is a flux of heat energy across each cross section. Let $q(x,t)$ denote the heat flux (amount of heat flowing) though the cross-section at x at time t (see Figure 4).
By definition the rate of change of heat energy with respect to time contained in $x_1 < x < x_2$ is

$$\frac{d}{dt}\int_{x_1}^{x_2} A\rho C_v u(x,t)dx.$$

By conservation of heat energy we have

$$\frac{d}{dt}\int_{x_1}^{x_2} A\rho C_v u(x,t)dx = q(x_1,t) - q(x_2,t).$$

which we can rewrite as

$$\frac{d}{dt}\int_{x_1}^{x_2} A\rho C_v u(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial q(x,t)}{\partial x}dx. \tag{1.23}$$

Since **Fourier's law** tells us that

$$q(x,t) = -KA\frac{\partial u}{\partial x}(x,t),$$

where K is the thermal conductivity (amount of heat flowing across a unit length per unit degree) we have that

$$\int_{x_1}^{x_2} A\rho C_v \frac{\partial u}{\partial t}(x,t)dx = \int_{x_1}^{x_2} KA\frac{\partial^2 u}{\partial x^2}(x,t)dx,$$

$$\Rightarrow \int_{x_1}^{x_2}\left(\frac{\partial u}{\partial t}(x,t) - \kappa\frac{\partial^2 u}{\partial x^2}(x,t)\right)dx = 0 \quad \text{where } \kappa = \frac{K}{\rho C_v}. \tag{1.24}$$

Equation (1.24) holds for all intervals (x_1, x_2) in $(0,l)$.

Lemma 1.1 (DuBois - Raymond)**.** *Let f be continuous on $(0,l)$. Suppose for all intervals (x_1,x_2) contained in $(0,l)$ that*

$$\int_{x_1}^{x_2} f(x)dx = 0.$$

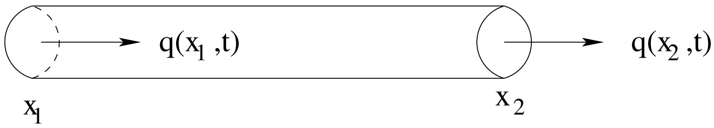

Figure 4: Heat flux in a rod

It follows that $f(x) = 0, \forall x \in (0, l)$.

Hence by DuBois - Raymond, we have

$$\frac{\partial u}{\partial t}(x,t) - \kappa \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \qquad \forall\, x \in (0,l). \tag{1.25}$$

Equation (1.25) is the heat equation. It is one of the most important (and ubiquitous) equations in mathematics, in it $\kappa = K/(\rho C_v)$ is the thermal diffusivity.
The heat equation is a partial differential equation for the function $u(x,t)$. We consider the case where (1.25) holds in the space time region $0 < x < l, t > 0$. Let us suppose that we have a situation where the temperature of the rod is initially zero, i.e.,

$$u(x,0) = 0, \qquad \forall\, x \in [0,l].$$

Suppose we now raise the temperature at the ends to be $u_0 > 0$. This gives us an initial boundary value problem for the unknown temperature $u(x,t)$.
In this initial boundary value problem we have the variables u, x and t and the parameters κ, l and u_0 with

	variables			parameters		
	u	x	t	l	u_0	κ
Dimension	Θ	L	T	L	Θ	L^2T^{-1}

Note that the fact that $[\kappa] = L^2T^{-1}$ can be deduced from the dimensionally correct partial differential equation $u_t = \kappa u_{xx}$:

$$\begin{aligned} [\kappa] &= [u_t/u_{xx}] \\ &= [u_t]/[u_{xx}] \\ &= \frac{\Theta T^{-1}}{\Theta/L^2} = L^2T^{-1}. \end{aligned}$$

Example 1.12 (Dimensionless form of the heat equation)**.**
We have derived the following initial boundary value problem for the flow of heat in a rod:
Find (x,t) *such that*

$$\begin{aligned} &u_t = \kappa u_{xx},\ 0 < x < l, t > 0, \\ &u(0,t) = u_0,\ u(l,t) = u_0,\ t > 0, \\ &u(x,0) = 0,\ 0 \le x \le l, \end{aligned}$$

where u_0 *is a given positive constant.*
Choose characteristic values for the variables in order to write the heat equation in dimensionless form.

Solution: In this case, it is natural to choose

$$u_c = u_0,\ x_c = l,\ t_c = l^2/\kappa$$

giving the dimensionless variables

$$\bar{u} = \frac{u}{u_0},\ \bar{x} = \frac{x}{l},\ \bar{t} = \frac{t\kappa}{l^2} \ \Rightarrow\ u = u_0\bar{u},\ x = l\bar{x},\ t = \frac{l^2}{\kappa}\bar{t}.$$

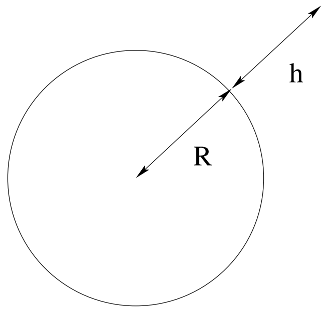

Figure 5: Projectile problem.

Under this change of variables we have $u(x,t) \to \bar{u}(\bar{x},\bar{t})$.
Using the chain rule (for example) we can derive the problem satisfied by $\bar{u} = \bar{u}(\bar{x},\bar{t})$. Since

$$u = u_0\bar{u} \;\Rightarrow\; \frac{\partial u}{\partial t} = u_0\frac{\partial \bar{u}}{\partial t} = u_0\frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = \frac{u_0 k}{l^2}\frac{\partial \bar{u}}{\partial \bar{t}}.$$

Similarly,

$$\frac{\partial u}{\partial x} = u_0\frac{\partial \bar{u}}{\partial x} = u_0\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} = \frac{u_0}{l}\frac{\partial \bar{u}}{\partial \bar{x}}.$$

A second differentiation gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_0}{l^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}.$$

Initial/boundary conditions become

$$\begin{cases} u_0\bar{u}(\bar{x},0) = 0 \\ u_0\bar{u}(0,\bar{t}) = u_0\bar{u}(1,\bar{t}) = u_0 \end{cases} \Leftrightarrow \begin{cases} \bar{u}(\bar{x},0) = 0 \\ \bar{u}(0,\bar{t}) = u_0\bar{u}(1,\bar{t}) = 1 \end{cases}$$

Remark: Alternatively we could use

$$\frac{\partial u}{\partial t} = \frac{\partial(u_0\bar{u})}{\partial(\kappa/l^2\bar{t})} = \frac{u_0}{l^2/\kappa}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\kappa u_0}{l^2}\bar{u}_{\bar{t}}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2(u_0\bar{u})}{\partial(l^2\bar{x}^2)} = \frac{u_0}{l^2}\frac{\partial \bar{u}}{\partial \bar{x}^2} = \frac{u_0}{l^2}\bar{u}_{\bar{x}\bar{x}}$$

Thus we have the following dimensionless form of the heat equation

$$\bar{u}_{\bar{t}}(\bar{x},\bar{t}) - \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = 0,$$
$$\bar{x} \in (0,1),\ \bar{t} > 0,$$

with boundary conditions

$$\bar{u}(0,\bar{t}) = \bar{u}(1,\bar{t}) = 1$$

and initial conditions

$$\bar{u}(\bar{x},0) = 0.$$

This is one problem for $\bar{u}$; solving it yields the solution to our original problem for any κ, l, u_0 by transforming back into its original variables.

Example 1.13. *We consider a projectile launched vertically upwarding from the earth's surface with the assumptions that air resistance and the rotation of the earth are neglected. The equation of motion for the projectile is given by*

$$\ddot{h}(t) = \frac{-R^2 g}{(R+h)^2}. \tag{1.26}$$

with initial data

$$h(0) = 0, \quad \dot{h}(0) = V > 0. \tag{1.27}$$

Here (see Figure 5)

R	*radius of earth*
g	*gravity*
$h(t)$	*height of projectile at time t.*

Choose characteristic value for the variables so that the problem can be written in terms of the dimensionless variables $\bar{h} = h/h_c$ and $\bar{t} = t/t_c$ and one dimensionless constant ε.

Solution:
We have the following variables and parameters: $h,\ t,\ R,\ g$ and V

	Variables		Parameters		
	h	t	R	g	V
Dimensions	L	T	L	LT^{-2}	LT^{-2}

We now choose characteristic value for the variables.
Version A

$$h_c = R \quad \Rightarrow \bar{h} = h/h_c = h/R$$
$$t_c = R/V \quad \Rightarrow \bar{t} = t/t_c = Vt/R.$$

The problem becomes:

$$\varepsilon \ddot{\bar{h}}(\bar{t}) = \frac{-1}{(1+\bar{h})^2},\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = 1,$$

where $\varepsilon = V^2/gR$ is dimensionless.

Version B

$$h_c = R,\ t_c = (R/g)^{1/2} \quad \Rightarrow \bar{h} = h/R,\ \bar{t} = t/(R/g)^{1/2}.$$

The problem becomes:

$$\ddot{\bar{h}}(\bar{t}) = \frac{-1}{(1+\bar{h})^2},\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = \sqrt{\varepsilon},$$

where $\varepsilon = V^2/gR$ is dimensionless.

Version C

$$h_c = \frac{V^2}{g},\ t_c = \frac{V}{g} \quad \Rightarrow \bar{h} = hg/V^2,\ \bar{t} = tg/V).$$

The problem becomes:

$$\ddot{\bar{h}}(t) = \frac{-1}{(1+\varepsilon\bar{h})^2},\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = 1,$$

where $\varepsilon = V^2/gR$ is dimensionless.

These three versions are all examples of scaling. Is any one better or more appropriate than the others? Is one of them correct?
The answer depends on the values of the parameters we are interested in and what problems we wish to answer.
Suppose we are interested in motion near the earth surface, this is associated with a small initial velocity V. Then $\varepsilon = V^2/(gR)$ is assumed to be small. So neglecting ε, i.e., putting it equal to zero in our three versions should give an approximation to the problem we wish to solve.
Approximation A

$$0 = \frac{-1}{(1+\bar{h})^2};\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = 1,$$

(does not make sense).

Approximation B

$$\ddot{\bar{h}} = \frac{-1}{(1+\bar{h})^2};\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = 0,$$

(cannot be right for our purposes as $\bar{h}$ becomes negative immediately.)

Approximation C

$$\ddot{\bar{h}}(t) = -1;\ \bar{h}(0) = 0,\ \dot{\bar{h}}(0) = 1.$$

$$\Rightarrow\ \bar{h}(\bar{t}) = \bar{t} - \frac{1}{2}\bar{t}^2.$$

Use $h = h_c\bar{h},\ t = t_c\bar{t}$.

$$\Rightarrow h(t) = Vt - \frac{1}{2}gt^2, \tag{1.28}$$

in original physical variables.
(1.28) is the standard formula for vertical particle motion assuming that acceleration due to gravity is constant near the earth surface.
The scaling which leads to Approximation C is the most appropriate for small positive initial velocity if we take $\varepsilon = V^2/(gR)$ as an indicator for the smallness of V.
What could have made us choose this scaling in the first place?
Since we are interested in the problem when is V small, we note that h will also be small and so if we solve the original problem with $h = 0$, i.e. if we solve

$$\ddot{h} = -\frac{R^2 g}{R^2 + 0} = -g,\quad \dot{h}(0) = V,\ h(0) = 0$$

we obtain

$$\dot{h} = -gt + V \quad \text{and} \quad h = -\frac{1}{2}gt^2 + Vt. \tag{1.29}$$

If we want to find scales h and t will vary on then we see from (1.29) that the maximum height reached in our approximate model occurs when $\dot{h} = 0$ i.e. when $t = V/g$ and that the maximum height is $h = \frac{1}{2}V^2/(gR)$.
Thus taking t_c to be the time that the maximum height is achieved and h_c to be twice the maximum height are natural and appropriate characteristic values with which to scale t and h in this setting.

Example 1.14. *Explain how, by using the Buckingham Pi Theorem, we could conclude that any dimensionless version of the projectile problem*

$$\ddot{h}(t) = \frac{-R^2 g}{(R+h)^2},$$
$$h(0) = 0,\ \dot{h}(0) = V.$$

would involve just one dimensionless parameter.

Solution:
The projectile problem involves the variables and parameters $h,\ t,\ R,\ g,\ V$ and has dimension matrix A given by

$$\begin{matrix} & h & t & R & g & V \\ L & 1 & 0 & 1 & 1 & 1 \\ T & 0 & 1 & 0 & -2 & -1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 \end{pmatrix} = A.$$

Clearly rank $A = 2$, so the Buckingham Pi Theorem implies that any law connecting the 5 parameters/variables would involve $5 - 2 = 3$ dimensionless quantities. Two of these are accounted for by the dimensionless variables $\bar{h}$ and $\bar{t}$, leaving one dimensionless parameter.

1.6 Remarks on Scaling

General idea : Work with dimensionless variables and coefficients, obtained by selecting appropriate characteristic values so as to obtain a simplification of the problem. (The Buckingham Pi theorem gives the total number of variables and coefficients involved.) Correct or appropriate scaling satisfies the following criteria :-

(a) Characteristic values are truly characteristic. That is each q_c is a typical or representative value of the corresponding variable q so that the scaled variable $\bar{q} = q/q_c$ is of 'moderate' size.

(b) Any dimensionless coefficient appearing in a term should give the size of the term in which it appears. The quantity it multiplies should have moderate size.

Remark 1.6.

(i) The notion of 'moderate size' can be made more precise in terms of orders of magnitude. (See Lin/Segel for a discussion.)

(ii) The point of (b) is that by looking at a term in the equation we should be able to estimate its size simply by looking at the coefficient and not have to consider the size of the variables in that term.

(iii) A 'correct' scaling satisfying (a) and (b) may be valid only for limited ranges of the variables and parameters involved. Other ranges may require other scaling.

(iv) Suppose we have a 'correct' scaling with t_c as a characteristic time. We refer to t_c as the time scale. A multi-time scale problem is one that requires different time scales for different ranges of time in order to achieve appropriate scaling. (Similar terminology is used for other variables.)

Example 1.15. *Consider the motion in an almost linear resistive medium, with equation of motion given by:*

$$m\frac{dV}{d\tau} = -aV + bV^2, \; V(0) = V_0. \tag{1.30}$$

Here

$$\begin{aligned} m &= \textit{mass of object}, \\ V &= \textit{velocity of object}, \\ \tau &= \textit{time} \end{aligned}$$

and we suppose that a and b are constants with $a > 0$ and $b \geq 0$ "small". Find suitable characteristic values V_c and τ_c so that the equation can be written in terms of the dimensionless variables $u = V/V_c$ and $t = \tau/\tau_c$ and one dimensionless constant ε.

Solution:
First we rewrite the equation as

$$\frac{dV}{d\tau} = \frac{a}{m}\left(-V + \frac{bV^2}{a}\right) = f(V)$$

and we note that we have the following variables and parameters:

variables/parameters	V	τ	m	a	b	V_0
Dimensions	LT^{-1}	T	M	MT^{-1}	ML^{-1}	LT^{-1}

Clearly a characteristic value for the velocity is V_0 and hence our new dimensionless variable u is given by

$$u = \frac{V}{V_0}.$$

What about the characteristic time?
We observe that (since b is small) we can solve an approximation to the problem by setting $b = 0$ giving:

$$\frac{dV}{d\tau} = -\frac{a}{m}V \quad \Rightarrow \quad \ln V = -\frac{a}{m}\tau + C.$$

The initial data $V(0) = V_0$ gives $C = \ln V_0$ and hence we have

$$\ln\left(\frac{V}{V_0}\right) = -\frac{a}{m}\tau \;\Rightarrow\; V = V_0 e^{-a\tau/m}.$$

The half life of V is the time τ^* such that

$$V(\tau^*) = \frac{1}{2}V_0 \quad \Rightarrow\; \tau^* = \frac{m}{a}\ln 2.$$

Hence a characteristic time scale for this problem is $\tau_c = m/a$.
So, our dimensionless time variable is

$$t = \frac{\tau}{\tau_c} = \frac{\tau}{m/a} = \frac{a}{m}\tau.$$

We proceed to write the equation in terms of u and t,

$$\begin{aligned}
\frac{d(V_0 u)}{d(m\tau/a)} &= \frac{a}{m}\left(-V_0 u + \frac{b}{a}(V_0 u)^2\right), \\
\Rightarrow \frac{aV_0}{m}\frac{du}{dt} &= \frac{a}{m}V_0\left(-u + \frac{b}{a}V_0 u^2\right) \\
\Rightarrow \frac{du}{dt} &= -u + \varepsilon u^2.
\end{aligned}$$

where $\varepsilon = V_0 b/a$ is small dimensionless constant. For the initial data we have

$$V_0 u(0) = V_0 \;\Rightarrow u(0) = 1.$$

Thus we have the rescaled problem

$$\frac{du}{dt} = -u + \varepsilon u^2, \quad u(0) = \frac{V_0}{V_0} = 1. \tag{1.31}$$

1.7 Scaling Using Known Functions

Suppose we wish to scale $u(x,t)$ in a problem where we are given a function $f(t)$ such that $[f(t)] = [u]$ for $t \in I$ where I is some interval appropriate to the problem under consideration. (For example suppose that u is the water temperature in a lake and $f(t)$ is some function that gives the average daily water temperature at a given point in the lake over the previous year).
Suppose f is bounded on I. A standard way of scaling f or u is to take

$$u_c = \sup_{t\in I}|f(t)|$$

and to introduce $\bar{u} = u/u_c$.
Although the length of I would seem to give an appropriate scale t_c, this is not always the case (e.g. if $I = \mathbb{R}$). However if f is differentiable on I with $f'(t)$ being bounded on I, then we can also obtain a characteristic value t_c for t in the following way:

$$t_c := \frac{u_c}{\sup_{t\in I}|f'(t)|} = \frac{\sup_{t\in I}|f(t)|}{\sup_{t\in I}|f'(t)|}$$

here t_c gives a time interval over which 'f' changes 'the most'.

Remark 1.7. *When choosing 'u_c' and 't_c' we don't have to find exactly the supremum of f and f', approximate values will do.*

Example 1.16. *Find u_c and t_c for $f(t) = A\sin\lambda t,\ t \in \mathbb{R}$ where A and λ are positive constants.*

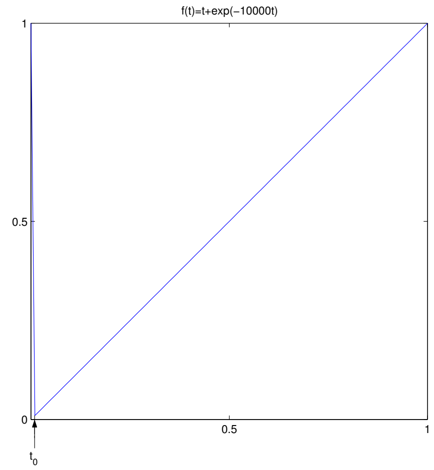

Figure 6: Example 1.17

Solution:
We have

$$u_c = \sup_{t\in\mathbb{R}} |A \sin \lambda t| = A$$

and

$$t_c = \frac{u_c}{\sup_{t\in\mathbb{R}} |f'(t)|} = \frac{u_c}{\sup_{t\in\mathbb{R}} |A\lambda \cos \lambda t|} = \frac{A}{A\lambda} = \frac{1}{\lambda}.$$

Note that it might be more convenient to use $t_c = 2\pi/\lambda$ period.

Example 1.17 (Multi-scale)**.** *Find u_c and t_c for*

$$f(t) = t + e^{-10000t}, \ t \in [0, 1].$$

Solution: We have

$$f'(t) = 1 - 10000e^{-10000t}.$$

The discussion above suggests

$$u_c = \sup_{t\in[0,1]} |f(t)| \approx 1,$$

so take $u_c = 1$.
For t_c the suggestion is

$$t_c = \frac{u_c}{\sup_{t\in[0,1]} |f'(t)|} = \frac{1}{9999}.$$

However this short time scale $t_c \approx 10^{-4}$ is only appropriate only near $t = 0$, where $f(t)$ changes rapidly with t (see Figure 6). The minimum value of f occurs when $t = t_0$ given by

$$0 = 1 - 10000e^{-10000t_0},$$

i.e.

$$t_0 \approx 0.00092 \approx 9t_c.$$

Using $\bar{t} = t/t_c$ changes the interval $[0, t_0]$ to $[0, 9]$, however the full interval $[0, 1]$ is changed into an interval with length $\approx 10^4$. Hence there are two time scales in the problem.
If we restrict our analysis to the interval $t \in [t_0, 1]$ then $u_c \approx 1$ and $t_c \approx 1$ as

$$\sup_{t\in[t_0,1]} |f(t)| \approx 1 \text{ and } \sup_{t\in[t_0,1]} |f'(t)| \approx 1$$

suggesting $u_c = 1$ and $t_c = 1$. So we should use the scaling :-

$$\bar{u} = \frac{u}{u_c}, \ \bar{t} = \frac{t}{t_c} \quad \text{where in } [0, t_0] \ \ u_c = 1, \ t_c = 10^{-4} \ \text{ and in } [t_0, 1] \ \ u_c = 1, \ t_c = 1.$$