Methods of Applied Mathematics (840G1)
Homogeneous and non-homogeneous ordinary differential equations with constant coefficients

A linear homogeneous ordinary differential equation with constant coefficients has the general form of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{'} + a_0 y = 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all constants.

1 Homogeneous equations (2nd order)

A second order linear homogenous ordinary differential equation with constant coefficients can be expressed as

$$a_2 y^{''} + a_1 y^{'} + a_0 y = 0$$

This equation implies that the solution is a function whose derivatives keep the same form as the function itself and do not explicitly contain the independent variable, since constant coefficients are not capable of correcting any irregular formats or extra variables. An elementary function which satisfies this restriction is the exponential function $e^{\lambda t}$.

Substitute the exponential function $e^{\lambda t}$ into the above differential equation, the **characteristic equation** of this differential equation is obtained

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

This characteristic equation has two roots λ_1 and λ_2.

	Solutions of characteristic equation	**General Solution**
1	$\lambda_1, \lambda_2 \in \mathrm{R}, \lambda_1 \neq \lambda_2$	$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
2	$\lambda_1, \lambda_2 \in \mathrm{R}, \lambda_1 = \lambda_2 = \lambda$	$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$
3	$\lambda_1 = a + ib, \lambda_2 = a - ib$, where $a, b \in \mathrm{R}$	$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = D_1 e^{ax} \cos(bx) + D_2 e^{ax} \sin(bx)$

2 Non-homogeneous equations

A linear non-homogeneous ordinary differential equation with constant coefficients has the general form of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{'} + a_0 y = r(x)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all constants and $r(x) \neq 0$.

For a linear non-homogeneous differential equation, the general solution is the superposition of the particular solution $y_p(x)$ and the complementary solution $y_c(x)$

$$y(x) = y_p(x) + y_c(x)$$

The complementary solution $y_c(x)$ is the general solution of the associated homogeneous equation ($r(x) = 0$) discussed in the previous section. For determining the particular solution $y_p(x)$, the two most commonly used methods are: **Method of undetermined coefficients** and **Method of variations of parameters.**

The non-homogeneous term $r(x)$ in a linear non-homogeneous ODE sometimes contains only linear combinations or products of some simple functions whose derivatives are more predictable or well known. By understanding these simple functions and their derivatives, we can guess the trial solution with undetermined coefficients, plug into the equation, and then solve for the unknown coefficients to obtain the particular solution. This method is known as the **Method of undetermined coefficients.** Below is a table giving trial functions for the particular solution based on the non-homogeneous term $r(x)$.

Format of non-homogeneous term $r(x)$	**Trial function for particular solution**
$k(constant)$	$c(constant)$
kx^m	$c_m x^m + c_{m-1}x^{m-1} + \ldots + c_1 x + c_0$
$ke^{\gamma x}$	$ce^{\gamma x}$
$k\cos(\alpha x + \beta)$ or $k\sin(\alpha x + \beta)$	$c_1\cos(\alpha x + \beta) + c_2\sin(\alpha x + \beta)$
$kx^m e^{\gamma x}$	$(c_m x^m + c_{m-1}x^{m-1} + \ldots + c_1 x + c_0)e^{\gamma x}$
$kx^m\cos(\alpha x + \beta)$ $kx^m\sin(\alpha x + \beta)$	$(c_m x^m + c_{m-1}x^{m-1} + \ldots + c_1 x + c_0)\cos(\alpha x + \beta)+$ $(d_m x^m + d_{m-1}x^{m-1} + \ldots + d_1 x + d_0)\sin(\alpha x + \beta)$
$ke^{\gamma x}\cos(\alpha x + \beta)$ or $ke^{\gamma x}\sin(\alpha x + \beta)$	$e^{\gamma x}[c_1\cos(\alpha x + \beta) + c_2\sin(\alpha x + \beta)]$
$(\sum_{i=0}^{n} a_i x^i)e^{\gamma x}\cos(\alpha x + \beta)$ or $(\sum_{i=0}^{n} b_i x^i)e^{\gamma x}\sin(\alpha x + \beta)$	$e^{\gamma x}[(\sum_{i=0}^{n} p_i x^i)\cos(\alpha x + \beta)+$ $(\sum_{i=0}^{n} q_i x^i)\sin(\alpha x + \beta)]$

Remark: The above table holds only when NO term in the trial function shows up in the complementary solution. If any term in the trial function does appear in the complementary solution, the trial function should be multiplied by x to make the particular solution linearly independent from the complementary solution. If the modified trial function still has common terms with the complementary solution, multiply by another x until no common term exists.

If the non-homogeneous term $r(x)$ is a sum of terms in the above table, the particular solution can be guessed using a sum of the corresponding trial functions.