Renewal Processes

4.1 Gary plays his aunt at draughts. For each game, the probability that Gary wins is p; the probability that his aunt wins, or the game is drawn, is thus $1-p$. Games are mutually independent. A renewal occurs if Gary wins two games running, and if a renewal did not occur on the first of these two games.

(a) Following the usual notation, write down the probabilities u_0, u_1, u_2 and f_1, f_2, f_3, f_4.

(b) Show that $u_n = (u_{n-2} + 1 - p)p^2$ for $n \geq 3$, and hence find the generating function $U(s) = \sum_{n=0}^{\infty} u_n s^n$.

(c) Find the probability generating function $F(s) = \sum_{n=1}^{\infty} f_n s^n$. By differentiating this function show that the expectation of the number of games from one renewal to the next is $(1+p)/p^2$.

(d) After a long sequence of games, Gary's mother arrives to find a game in progress. What is the probability that it will end in a renewal?

4.2 Suppose that the lifetime T of a component has distribution given by $P(T=1) = P(T=2) = P(T=3) = 1/3$.

(a) Find the p.g.f. $F(s)$ of T, and thus the p.g.f. of W_r, the waiting time to the r^{th} failure.

(b) What is the probability that the 3^{rd} failure occurs at or after time 7?

(c) Find the probability that the 4^{th} failure occurs at time 6.

4.3 It is decided to model the lifetime of a spring using a uniform distribution on (0,1), time being measured in years.

(a) What is the distribution of the future lifetime, T_z, of a spring that has survived for a time z?

(b) Write down the mean and standard deviation of T_z.

(c) Prove that the spring is 'new better than used'.

(d) Find the hazard rate $h(t)$ corresponding to the lifetime distribution.

4.4 Consider a renewal process that has been running a long time, with lifetimes having the distribution $\text{Unif}(4, 16)$.

(a) Show that the mean lifetime of the component in use is $11{\cdot}2$, and deduce the mean time to the next renewal.

(b) Find the density of the time to the next renewal, and verify that its mean agrees with the value obtained in (a).

4.5 A renewal process has lifetimes with density

$$g(t) = \frac{2}{t^3} \qquad (t > 1).$$

We join the process after it has been running for some time. Find the following.

(a) The distribution of W, the lifetime of the component currently in operation.

(b) The distribution of the residual lifetime of the component in operation.

(c) The approximate expected number of renewals by time $t = 20$.