Birth and death processes

2.1 Consider a population, initially with 3 individuals, such that every individual gives birth at a rate of 1 per week on average.

(a) What is the probability that there are exactly 10 individuals after 2 weeks?

(b) How long do we have to wait before the probability that there are no new individuals is less than 0·01?

(c) Given that there are still 3 individuals after 1 week, and 5 individuals after 2 weeks, what is the probability that there are 10 individuals after 3 weeks?

2.2 Ten unemployed former students join a job club. As soon as one gets a job, he or she leaves. Suppose that each is equally likely to get a job, and receives offers (which are always accepted) at average rate 3 per year.

(a) What is the probability that all the students have got a job after 2 years?

(b) What is the probability that the last student to get a job gets it in the third year?

(c) How long must we wait before the probability that all the students have accepted a job is greater than 0·5?

2.3 What are the Kolmogorov forward equations for

(a) the pure death process?

(b) a birth-death process with death rate 1 and birth rate $2/x$ (for $x \geq 1$), where x is the population size?

2.4 Suppose that a population of animals is such that, on average, individuals give birth at rate 4 per year, but also die at rate 3·8 per year. The initial population is of size 10. What is the probability that

(a) the population becomes extinct?

(b) the population is of size 1 or 0 after 18 months?

(c) exactly 2 out of the first 5 birth/death events are births?

(d) the population reaches 1, then reaches at least 10, but then becomes extinct?

2.5 A population follows an immigration-death process, where the immigration and death rates are $\lambda = 2$, $\nu = 1$.

(a) Give the equilibrium distribution and thus find the probability that there are more than 2 but less than 6 individuals in the population.

(b) Suppose that a second population independently follows an immigration-death process with rates $\lambda = 24$, $\nu = 2$. What is the probability that the total number of individuals from the two populations is 17?

2.6 In a small village, the car park is a lay-by with four spaces. Shoppers arrive to park their car at average rate of 6 an hour. A shopper takes a free space, but otherwise drives away. The length of stay of any shopper is exponential, mean half an hour, independently of the other shoppers.

(a) Find the Kolmogorov equations for this situation and hence show that the equilibrium distribution for N, the number of parked cars, is given by

$$P(N = n) = \frac{8}{131}\frac{3^n}{n!} \qquad (n = 0, 1, 2, 3, 4).$$

continued ...

(b) For what proportion of the time does an arriving driver find no available space?