Poisson processes

1.1 Let X have the exponential distribution with density $\lambda e^{-\lambda x}$ for $x > 0$. Show that X has the lack of memory property, i.e. $P(X > s + t | X > s) = P(X > t)$. Sketch the density and distribution function of X, and show that $EX = 1/\lambda$, $\operatorname{var} X = 1/\lambda^2$.

1.2 Cars pass a certain tree on a quiet country road at an average rate of one every two minutes.

(a) What is the probability that exactly five cars pass the tree in ten minutes?
(b) What is the probability that exactly five cars pass the tree in ten minutes, given that in the first five of these minutes one car passes the tree?
(c) What is the probability that in all of the periods 10.00–10.02, 10.02–10.04, 10.04–10.06, 10.06–10.08, 10.08–10.10, at least one car passes the tree?
(d) Would a Poisson process be a good model for a busy road? Explain.

1.3 Given a Poisson process of constant rate λ, consider each of the following alterations in turn:

(a) Delete every alternate point.
(b) Insert points at times 1, 2, 3, ...
(c) Insert points at times $2t$, whenever there is a point at t.
(d) Delete points that occur in intervals $(3k - 1, 3k]$, for $k = 1, 2, 3, \ldots$.
(e) Insert points from an independent Poisson process, rate μ.
(f) Delete points that occur within a fixed time of other points.
(g) For each point delete it with probability p, independently of other points.

Which of the above are Poisson processes? Why? And for those that are, what is the rate?

1.4 A certain large pub has three entrances: the front, the back and the side doors. The landlord thinks that during lunchtime customers arrive at rate one per minute through the front door, one every two minutes through the back door and one every three minutes through the side door, and models this by a Poisson process for each entrance. The pub opens at 12.00. What is the probability of each of the following events?

(a) No customers arrive through the front door by 12.06.
(b) No customers arrive through any door by 12.06.
(c) Exactly five customers arrive in the pub by 12.06, given that exactly two came through the front door.
(d) At least two customers enter by each door by 12.06.
(e) The second customer to enter the pub comes in through the back door.

1.5 In another pub, orders for drinks occur as a Poisson process of rate 1 per minute, and the landlord notices that the proportions of beer/wine/spirits among the orders are 0·6/0·15/0·25 respectively.

(a) How long must he wait before there is a probability of at least 0·8 that he has served a beer (ignoring service time)?
(b) What is the probability that the first three orders are for the same type of drink as each other?
(c) Find approximately the probability that there are more orders for spirits than for wine in the first two minutes.