used.

Rescaling in the expansion method

Instead of starting the expansion with the unusual ϵ^{-1} term, a very useful idea for singular problems is to rescale the variables before making the expansion. Thus introducing the rescaling

$$x = X/\epsilon$$

into the originally singular equation for x produces an equation for X,

$$X^2 + X - \epsilon = 0$$

which is regular. Thus the problem of finding the correct starting point for the expansion can be viewed as a problem of finding a suitable rescaling to regularise the singular problem.

There is a simple procedure to find all useful rescalings. First one poses a general rescaling with a scaling factor $\delta(\epsilon)$,

$$x = \delta X$$

in which one insists that X is strictly of order unity as $\epsilon \to 0$. Unfortunately the standard notation $X = O(1)$ does not describe this limitation on X, because $O(1)$ permits X to be vanishingly small as $\epsilon \to 0$. Thus we are forced to adopt the less familiar notation $X = \text{ord}(1)$ to stand for X *is strictly of order unity as* $\epsilon \to 0$.

Substituting the general rescaling into the governing quadratic equation gives

$$\epsilon\delta^2 X^2 + \delta X - 1 = 0$$

We now consider the dominant balance of this equation for δ of different magnitudes, starting the search for sensible rescalings with δ very small and progressing to δ very large.

- $\delta \ll 1$. If δ is very small, then the left hand side of the equation is

$$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + \text{small} - 1$$

This cannot balance the zero on the right hand side, and so a small δ is an unacceptable rescaling. As δ is increased, it is the X term which first breaks the domination of the 1 term and this occurs when $\delta = 1$. Hence the range of the unacceptable rescalings when δ is too small is $\delta \ll 1$, as declared above.

- $\delta = 1$. The left hand side of the equation is now

$$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + X - 1$$

This can balance the zero on the right hand side to produce the regular root $X = -1 + \text{small}$.

- $1 \ll \delta \ll \epsilon^{-1}$. If δ is a little larger than unity, then the X term dominates the left hand side of the equation

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\delta = \text{small} + X + \text{small}$$

This can balance the zero divided by δ on the right hand side, but only if $X = 0 + \text{small}$ which violates the restriction that X is strictly of order unity and not smaller. This rescaling is therefore unacceptable. As δ increases well beyond unity, it is the X^2 term which breaks the domination of the X term when $\delta = \epsilon^{-1}$. Hence this range of unacceptable rescalings is $1 \ll \delta \ll \epsilon^{-1}$, as declared above.

- $\delta = \epsilon^{-1}$. The left hand side of the equation divided by $\epsilon\delta^2$ is

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\epsilon\delta^2 = X^2 + X + \text{small}$$

This can balance the zero divided by $\epsilon\delta^2$ on the right hand side with either $X = -1 + \text{small}$ which yields the singular root, or $X = 0 + \text{small}$ which is not permitted because it violates our restriction $X = \text{ord}(1)$.

- $\epsilon^{-1} \ll \delta$. Finally when δ is very large, the left hand side of the quadratic divided by $\epsilon\delta^2$ is

$$(\epsilon\delta^2 X^2 + \delta X - 1)/\epsilon\delta^2 = X^2 + \text{small} + \text{small}$$

This can only balance the right hand side if $X = 0{+}\text{small}$, which violates $X = \text{ord}(1)$. Thus $\epsilon^{-1} \ll \delta$ is a range of unacceptable rescalings.

The systematic search of all possible rescalings has thus yielded $\delta = 1$ for the regular root and $\delta = \epsilon^{-1}$ for the singular root as the only possible rescalings with $X = \text{ord}(1)$.